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Research Article

Ridge and Liu-Type Estimators for Tobit SUR Model: Application to Air Pollution Data

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Abstract

This study extends the seemingly unrelated regression (SUR) model through introducing the ridge (RTSUR) and Liu-Type (LTSUR) estimators as biased estimation techniques to address the problem of multicollinearity in the SUR Tobit (SURT) model. This study theoretically evaluates the superiority of the proposed estimators based on the mean square error (MSE) criterion. The results for the theoretically study showing that, the Liu-Type estimator outperforms other estimators under many conditions. A simulation study was conducted to compare the estimators under various factors. The results of simulation show that, the maximum likelihood (MLE) estimator is the worst estimator at all factors and the LTSUR and RTSUR estimators perform better at high levels of multicollinearity and censoring. The LTSUR still achieved a significant superiority over the RTSUR. In addition, when the number of observations in the equations increases, the performance of the LTSUR and RTSUR estimators improves. Moreover, the simulation mean squared errors (SMSE) values for LTSUR estimator converge as number of observation and censored level increase. To study the behavior of the proposed estimators on real data, we used weather data from Cairo city to examine their influence on pollution levels of carbon monoxide (CO), sulfur dioxide (SO_2), and nitrogen dioxide (NO_2). The results from the real data were consistent with the findings from the simulation study.

Keywords: Liu-Type Tobit SUR estimator, multicollinearity, ridge Tobit SUR estimator, SUR Tobit model

1. INTRODUCTION

The SUR model, introduced by Zellner in 1962, is considered one of the important types of regression models. This model combines multiple regression equations with correlated error terms across them [1]. The best unbiased estimator for the SUR model is the ordinary least squares (OLS) estimator. However, in many cases, the SUR model exhibits high correlation between independent variables, leading to the problem of multicollinearity. In fact, when there is a very high level of multicollinearity, the hat matrix (X^TX) becomes ill-conditioned. Moreover, at high levels of multicollinearity, the OLS estimator becomes unsuitable because the variance of error terms become high, and lead to make misleading results Li and Yang (2012) [2]. Many studies have addressed multicollinearity in the SUR model. Those studies used penalized estimators to deal with this problem. Srivastava and Giles (1987) proposed a biased estimator known as the ridge SUR estimator (RE), which addresses the ill-conditioning by adding more information to the hat matrix [3]. The RE estimator depends on a single parameter, known as the ridge parameter. The crucial main of work efficiency of the RTSUR estimator is the select of the ridge parameter. Srivastava and Giles (1987), Firinguetti (1997), Kibria (2003), Omara (2018, 2019, 2025), Algamal (2018, 2019, 2020), and Abonazel et al. (2022) have explored various methods for selecting this parameter. The parameter in the ridge estimator is not constrained with certain height [3]-[13]. Alkhamisi (2008) suggested the ridge estimator for SUR model [14]. El-Houssainy et. al (2011) used cross validation method to select the ridge parameter in SUR model [15]. Therefore, Liu (2003) suggested a Liu-type estimator (LTE), which combines the ridge estimator with the stein estimator [16]. The LTE has two parameters that work together, with the increase of one controlled by the decrease of the other [17]. Wu (2014) later suggested the LTE for the SUR model [18]. The critical point in the work of the LTE is the selection of the estimation parameters. Omara (2021) used K-fold cross validation and robust formula to select Liu-type parameter [19].

On the other hand, in many cases, the dependent variable can have real zeros without missing values, or it may be constrained by upper or lower bounds.

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Table 1: The factors for the simulation study.

Factors	Values
Sample size per equation (n)	100, 500
Number of replications	1500
Number of equations (m)	4,10
Number of parameters per equation (p)	5,15
True coefficients	Chosen such that $\beta^T\beta = 1$
Multicollinearity levels (p_x)	0.85,0.90,0.95
Errors correlation (p_ε)	0.2,0.9
Censoring levels (ς)	5%, 20% ,40%

In these cases, the dependent variable must be censored. When the values of the dependent variable are censored, the OLS estimator becomes unstable and the MLE estimator can be employed as non-penalized estimator [20]. To take into account the presence of the dependent variable under censored, Tobin (1958) proposed a new non-linear regression model, known as the Tobit regression [21]. There are many studies that have applied the Topit model to various fields, such as Iqbal et al (2025) employed the Tobit regression model to evaluate the energy efficiency of buildings [22], while Moluno et al (2025) apply it in the agricultural field [23]. Similarly, Zhao et al. (2025) applied the Tobit regression model to determinants the efficiency of health resource allocation and in the same vein [24], Sun et al (2025) used Tobit regression model to analysis the efficiency of pharmaceutical companies [25].

Mai (2025) suggested a new Bayesian approach to high-dimensional Tobit regression [26]. Dănăilă and Buiu, (2024) applied advanced machine learning techniques, specifically deep learning methods, to address a censored regression problem [27]. To address multicollinearity in the Tobit model, Khalaf et al. (2014) used a ridge estimator [28]. Aydın et al (2020) study the performance of censored ridge estimator at optimal parameter [29]. Similarly, Alhusseini and Odah (2016) employed principal components methodology to handle ill-conditioning in Tobit regression [30]. Additionally, Toker et al. (2020) proposed a Liu estimator for the Tobit model [31], while Omara (2023) introduced an almost unbiased LTE for the Tobit model, which benefits from the properties of both the Liu-type and almost unbiased estimators [32]. Many studies have focused on combining the SUR and Tobit models to form the SUR Tobit (SURT) model. In fact, the SURT model emerged as a reduced form of a simultaneous equation Tobit model [33]. There are many penalized estimators for SURT model. In particular, Kamakura and Wedel (2001) suggested the MLE estimator for the SURT model [34]. Additionally, Huang et al. [35] and Huang [33] used an expectation-maximization (EM) algorithm for this purpose. Furthermore, studies such as Huang (2001), Baranchuk and Chib (2008), and Taylor and Phaneuf (2009) have explored the Bayesian analysis of the SURT model [36]-[38]. According to previous studies, we fond absence of penalized estimators to address multicollinearity in SUR Tobit regression models. In this paper, we provide a new methodological framework for stable and efficient estimation for SUR Tobit regression model. In addition, we introduce the RTSUR and LTSUR estimators where there has ability to stabilize estimation when the design matrix is ill-conditioned. These estimators can effectively address multicollinearity in the SUR Tobit model.

Herein, we evaluate the performance for the present estimator's theoretically using MSE criterion and empirically through run, the simulation studies under many factors. We also apply the proposed estimators to real data for examine pollution levels in Cairo city. Additionally, we utilize the MCECM (Monte Carlo Expectation Conditional Maximization) algorithm as introduced by Huang [33], to estimate the proposed estimators.

2. THE SUR TOBIT (SURT) MODEL

Consider the SURT model as shown in Equation

Table 2. The SMSE of MLE, RTSUR, and LTSUR under different n, p, ρ_ε and ρ_x with $\varsigma = 5\%$ and m = 4.

Estimators	$n = 100$											
	$p = 5$						$p = 15$					
	$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$			$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$		
	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$
MLE	33.6098	35.1879	39.3066	34.6654	36.6816	41.0112	36.3678	37.9023	40.9998	38.5643	42.0097	43.3164
RTSUR	18.1542	19.9755	22.4912	20.7821	23.2911	27.1215	21.5081	24.0989	28.5553	25.9067	27.7651	30.4823
LTSUR	11.2148	13.7816	14.0279	12.2581	14.4912	15.5463	12.5313	14.5460	15.9987	13.7692	16.6935	18.8921
Estimators	$n = 500$											
	$p = 5$						$p = 15$					
	$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$			$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$		
	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$
MLE	30.9892	32.9172	36.3412	32.619	32.6192	37.9132	32.9935	33.8973	35.5643	36.2953	39.2934	40.0022
RTSUR	13.3751	15.2753	18.9978	14.4421	17.7773	20.0182	15.2856	18.9064	22.2523	17.5612	20.0876	24.2351
LTSUR	9.6833	10.9271	11.0024	10.4421	12.1421	13.9974	11.4622	12.2671	13.9723	11.1178	13.1654	15.1442

Table 3. The SMSE of MLE, RTSUR, and LTSUR under different n, p, ρ_ε and ρ_x with $\varsigma = 5\%$ and m = 10.

Estimators	$n = 100$											
	$p = 5$						$p = 15$					
	$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$			$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$		
	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$
MLE	39.8953	42.2431	45.8761	41.1319	44.5424	48.1291	42.0092	45.7871	48.9095	43.5429	48.9803	52.8765
RTSUR	19.8921	20.0012	20.9145	20.4321	22.3263	25.3968	20.9997	23.3256	27.0561	26.3251	30.0254	32.3659
LTSUR	12.2529	15.3269	15.994	14.0328	16.0069	17.3181	15.9076	17.1564	18.0276	16.8691	18.8875	20.2557
Estimators	$n = 500$											
	$p = 5$						$p = 15$					
	$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$			$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$		
	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$
MLE	36.6325	40.3986	42.0654	37.6911	39.9571	42.9372	39.0675	42.1653	44.2979	41.3254	44.6942	48.8874
RTSUR	16.3672	17.6986	18.0035	17.365	19.3658	22.3654	18.7691	20.0036	24.6912	21.3648	24.691	26.8611
LTSUR	11.1934	13.3641	14.0367	12.2588	14.3901	15.5927	14.4891	15.5975	16.0947	15.5089	16.3927	17.0354

(1);

$$y^*_{ij} = x^T_{ij}\beta_i + \varepsilon_{ij}, i = 1, 2, \dots, m, j = 1, 2, \dots, n \quad \textbf{(1)}$$

$$y_{ij} = \begin{cases} y^*_{ij}, & if\ y^*_{ij} > 0 \\ 0, & if\ y^*_{ij} \le 0 \end{cases}, i = 1,2,\dots,m, j = 1,2,\dots,n$$

where m is the number of equations, n is the number of observation for each equation and y^*_{ij} is the latent dependent variable, x_{ij} is the j-th row of the n x $(p + 1)$ known matrix x_i, where p is the number of explanatory variables, β_i is $(p + 1)$ x 1 vector of coefficients and $\varepsilon_j=(\varepsilon_{1j}, \varepsilon_{2j}, \dots, \varepsilon_{mj})^T \sim N_m (0,\Sigma)$ is the vector of error terms and Σ is an m×m symmetric positive definite (p.d) matrix. The errors are independent across observations j. The variable y_{ij} equals to the true value of the dependent variable if $y^*_{ij} > 0$ and equals zero if $y^*_{ij} \le 0$. This assumption reflects a limited dependent variable where observations cannot fall below zero.

The SUR Tobit model has the normality and at the same time, the errors are correlated across equations, independent across observations, and censored at zero. These assumptions extend the classical Tobit model to a multivariate system where joint estimation improves efficiency compared to separate Tobit regressions.

The model expressed in Equation (2) can be rewrite as:

$$y^*_j = x_j\beta + \varepsilon_j, j = 1, 2, \dots, n \quad \textbf{(2)}$$

Where $y^*_j = (y^*_{1j}, y^*_{2j}, \dots, y^*_{mj})^T$, $X_j = blockdiag\ (x^T_{1j}, x^T_{2j}, \dots, x^T_{mj})$ and $\beta = (\beta_1^T, \beta_2^T, \dots, \beta_m^T)^T$.

From the model (1), since we have m equations, the censoring points have 2^m possible combinations. The 2^m possible combinations are represented by vectors S_h where $h = 1, 2, \dots, 2^m$, defined as:

$$S_1 = (0,0,\dots,0)^T, \dots, S_h = (0 \dots \underbrace{0, +}_{\mathrm{r}} \cdots \underbrace{+), \dots}_{\mathrm{m\text{-}r}}, S_{2^m} = (+,+,\dots,+)$$

where ,0, indicates that the observed value is non -positive (i.e, censored) and + indicates that the observed value is positive (i.e, uncensored).

Since the Tobit model is non-linear, the MLE is a candidate for estimating the model. The likelihood function (Equation (3)), based on all observations of the censoring combinations, is:

$$L(Y;\beta,\Sigma) = \prod_{j=1}^{n} L_j^{(S_h)}(y_j;\beta,\Sigma) \quad \textbf{(3)}$$

Where $Y = (y_1^T, y_2^T, \dots, y_n^T)^T$ and L^{Sh}_j is likelihood function for the j^{th} observation falling into regime h. Since the likelihood function is non-linear and intractable for all observations, it is can be solved iteratively. The EM algorithm is an iterative method that relies on the E-step (expectation step) and the M-step (maximization step). This method takes advantage of the simplicity of the likelihood function $f(Y, Y^*; \theta)$ and depends on the full data, both observe data Y and latent data Y^*. Let $\theta = (\beta, \Sigma)$. For the E-step, we compute Equation (4):

$$Q(\theta,\theta^{(i)}) = \int_{Y^*} lnf(Y,Y^*;\theta)p(Y^*;Y,\theta^{(i)})dY^*$$
$$= \int_{Y^*} lnf(Y^*;\theta)p(Y^*;Y,\theta^{(i)})dY^*, i = 1,2,\dots t \quad \textbf{(4)}$$

Where Y and Y^* are the observed and latent data, $\theta^{(i)}$ is value for θ at i iterative and t is number of iteration for the Q function. In the M-step, the Q function is maximized to obtain the iterative estimator $\theta^{(i+1)}$, $i = 1, 2, \dots, t$.

In fact, under EM algorithm there is a difficulty in calculating the multiple integrals of the function Q, making step E challenging. Huang (1999) suggested the Monte Carlo expectation conditional maximization (MCECM) algorithm, which uses the Monte Carlo simulation approach in the E-step [33]. The MCECM algorithm is used for estimation in models where the likelihood function is difficult to maximize directly. It involves alternating between updating latent variables via Markov chain methods and maximizing the expectation of the complete-data log-likelihood. From the MCECM algorithm, the sequence of samples $y^{*(a)}$, $a = 1, 2, \dots, N$ is generated, where N is the number of samples. In the second step, we calculate the Q function as Equation (5):

$$\hat{Q}(\theta,\theta^{(i)}) = \frac{1}{N}\sum_{a=1}^{N}\sum_{j=1}^{n} lnf(y_j^{*(a)};\theta)$$
$$= -\frac{nm}{2}\ln(2\pi) - \frac{n}{2}ln|\Sigma| + \frac{1}{N}\sum_{a=1}^{N}\left\{-\frac{1}{2}\sum_{j=1}^{n}(y_j^{*(a)} - X_j\beta)\Sigma^{-1}(y_j^{*(a)} - X_j\beta)\right\}, i$$
$$= 1.2 \dots . t \quad \textbf{(5)}$$

As $N \to \infty$, the $\hat{Q}(\theta,\theta^{(i)})$ function converges to $Q(\theta,\theta^{(i)})$..

The iterative estimators $\beta^{(i+1)}$ and $\Sigma^{(i+1)}$ are

Table 4. The SMSE of MLE, RTSUR, and LTSUR under different n, p, ρ_ε and ρ_x with $\varsigma = 20\%$ and m = 4.

Estimators	$n = 100$											
	$p = 5$						$p = 15$					
	$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$			$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$		
	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$
MLE	45.9354	48.3657	50.3927	47.325	49.905	53.967	49.1594	52.6548	54.9051	49.3997	54.0845	56.6985
RTSUR	24.6391	26.9287	28.8906	25.0694	28.9258	30.0054	27.9342	31.3697	33.3648	30.6582	35.5896	37.7894
LTSUR	15.0264	16.6942	17.3694	16.9302	18.3295	20.2364	18.8942	22.3628	25.3621	20.9026	24.3659	29.3254
Estimators	$n = 500$											
	$p = 5$						$p = 15$					
	$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$			$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$		
	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$
MLE	42.0258	44.0091	47.3658	43.3694	46.6358	50.6661	45.5296	48.9837	53.3654	46.6958	51.9784	53.6914
RTSUR	20.0364	23.3258	25.9285	21.0036	23.3698	25.5983	23.9866	26.6984	27.7846	25.9863	28.8945	31.3695
LTSUR	13.3695	14.2561	16.0002	14.4894	16.9481	17.9986	16.9872	18.0694	19.9975	18.6532	20.3205	22.3091

Table 5. The SMSE of MLE, RTSUR, and LTSUR under different n, p, ρ_ε and ρ_x with $\varsigma = 20\%$ and m = 10.

Estimators	$n = 100$											
	$p = 5$						$p = 15$					
	$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$			$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$		
	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$
MLE	49.6592	53.3694	55.9257	51.3264	53.3647	55.6241	54.3628	57.3256	60.6995	56.6941	58.6945	63.3666
RTSUR	26.6946	28.9874	30.3256	27.98644	31.3628	33.9958	29.2671	34.9625	38.8657	35.3698	38.8653	41.2578
LTSUR	17.8529	18.8751	20.3204	18.9457	21.6283	22.9025	19.9357	23.6387	27.3254	22.9831	25.5894	31.1579
Estimators	$n = 500$											
	$p = 5$						$p = 15$					
	$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$			$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$		
	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$
MLE	45.9681	47.0005	39.3658	46.3597	48.8898	51.1822	47.6924	50.5281	53.9987	49.9625	53.9286	56.9375
RTSUR	23.0055	25.3659	28.3654	24.9325	27.9864	29.9687	26.0204	29.9634	32.3692	28.8954	31.9998	34.6938
LTSUR	15.5963	17.0065	19.9265	16.6685	18.8958	21.0254	17.9833	20.3599	24.3628	19.9638	22.3628	27.3695

obtained by taking the first derivative of Equation (5) with respect to β and Σ respectively then setting the results equal to zero.

$$\frac{\partial \hat{Q}(\theta,\theta^{(i)})}{\partial \beta} = \sum_{a=1}^{N}\sum_{j=1}^{n} X_j^T \Sigma^{-1}\left(y_j^{*(a)} - X_j\beta\right) = 0$$

$$\frac{\partial \hat{Q}(\theta,\theta^{(i)})}{\partial \Sigma^{-1}} = -\Sigma + \frac{1}{N}\sum_{a=1}^{N}\sum_{j=1}^{n}\left(y_j^{*(a)} - X_j\beta\right)\left(y_j^{*(a)} - X_j\beta\right) = 0$$

The iterative estimators $\beta^{(i+1)}$ and $\Sigma^{(i+1)}$ are defined as Equations (6) and (7):

$$\beta^{(i+1)} = \beta^{(i)} + \left[\sum_{a=1}^{N}\sum_{j=1}^{n}\left(X_j^T\Sigma^{-1}X_j\right)\right]^{-1}\left[\sum_{a=1}^{N}\sum_{j=1}^{n}\left(X_j^T\Sigma^{-1}y_j^{*(a)}\right)\right], i = 1.2 \ldots\ldots t \quad \textbf{(6)}$$

$$\Sigma^{(i+1)} = \Sigma^{(i)} + \frac{1}{N}\sum_{a=1}^{N}\sum_{j=1}^{n}\left(y_j^{*(a)} - X_j\beta\right)\left(y_j^{*(a)} - X_j\beta\right)^T, i = 1.2 \ldots\ldots t \quad \textbf{(7)}$$

If we use the iteratively weighted least squares (IWLS), we obtain the maximum likelihood estimator (MLE) by solving Equations (6) and (7) iteratively. The MLE estimator is defined as Equation (8);

$$\hat{\beta}_{MLE}^{(i+1)} = \hat{\beta}_{MLE}^{(i)} + \left[\sum_{j=1}^{n} X_j^T\Sigma^{-1}X_j\right]^{-1}\sum_{j=1}^{n}\left[X_j^T\Sigma^{-1}Y_j^{*}\right] \quad \textbf{(8)}$$

where $Y_j^* = 1/N\, \Sigma^{N}_{a=1}\, y_j^{*(a)}$ and Σ^{1} is calculated by Equation (7). We can simplify the Equation (8) as:

$$\hat{\beta}_{MLE} = [X^T\Sigma^{-1}X]^{-1}X^T\Sigma^{-1}Y^{*}$$

The mean squared error (MSE) of an estimator is defined as Equation (9):

$$MSE(\hat{\beta}_{MLE}) = Var(\hat{\beta}_{MLE}) = \left[\Sigma_{j=1}^{n}\left(X_j^T\Sigma^{-1}X_j\right)\right]^{-1} = [X^T\Sigma^{-1}X]^{-1} = C^{-1} \quad \textbf{(9)}$$

3. THE RIDGE AND LTE FOR SUR TOBIT MODEL

For the model in Equation (1), we introduce RTSUR and LTSUR estimators, which are obtained by adding $k\sum_{s=1}^{m}\beta_s^T\beta_s$ and $\Sigma_{s=1}^{m}\left(k^{1/2}\beta_s - \frac{d\hat{\beta}_{MLE_s}}{k^{1/2}}\right)^T\left(k^{1/2}\beta_s - \frac{d\hat{\beta}_{MLE_s}}{k^{1/2}}\right)$ as conditions to Equation (5), where $k > 0$ and $0 < d < 1$ for all equations. Then RTSUR estimator can be defined as Equations (10) and (11):

$$\hat{Q}(\theta,\theta^{(i)}) = \frac{1}{N}\sum_{a=1}^{N}\sum_{j=1}^{n} lnf\left(y_j^{*(a)};\theta\right) + k\sum_{s=1}^{m}\beta_s^T\beta_s$$
$$= -\frac{nm}{2}ln|\Sigma| + \frac{1}{N}\sum_{a=1}^{N}\left\{-\frac{1}{2}\sum_{j=1}^{n}\left(y_j^{*(a)} - X_j\beta\right)^T\Sigma^{-1}\left(y_j^{*(a)} - X_j\beta\right)\right\} + k\sum_{s=1}^{m}\beta_s^T\beta_s \quad \textbf{(10)}$$

$$\hat{Q}(\theta,\theta^{(i)}) = \frac{1}{N}\sum_{a=1}^{N}\sum_{j=1}^{n} lnf\left(y_j^{*(a)};\theta\right) + \sum_{s=1}^{m}\left(k^{1/2}\beta_s - \frac{d\hat{\beta}_{MLE_s}}{k^{1/2}}\right)^T\left(k^{1/2}\beta_s - \frac{d\hat{\beta}_{MLE_s}}{k^{1/2}}\right)$$
$$= -\frac{nm}{2}ln|\Sigma|$$
$$+\frac{1}{N}\sum_{k=1}^{N}\left\{-\frac{1}{2}\sum_{j=1}^{n}\left(y_j^{*(a)} - X_j\beta\right)^T\Sigma^{-1}\left(y_j^{*(a)} - X_j\beta\right)\right\}$$
$$+\sum_{s=1}^{m}\left(k^{1/2}\beta_s - \frac{d\hat{\beta}_{MLE_s}}{k^{1/2}}\right)^T\left(k^{1/2}\beta_s - \frac{d\hat{\beta}_{MLE_s}}{k^{1/2}}\right) \quad \textbf{(11)}$$

The first derivative of Equations (10) and (11) with respect to β are obtained and set the results to zero then define $\beta^{(i+1)}$ after t iterations for RTSUR and LTSUR as

$$\beta^{(i+1)}{}_{RTSUR} = \beta^{(i)}{}_{RTSUR} + \left[\sum_{j=1}^{n}\left(X_j^T\Sigma^{-1}X_j\right) + kI\right]^{-1}\sum_{a=1}^{N}\sum_{j=1}^{n}\left(X_j^T\Sigma^{-1}y_j^{*(a)}\right), i = 1.2 \ldots t \quad \textbf{(12)}$$

$$\beta^{(i+1)}{}_{LTSUR} = \beta^{(i)}{}_{LTSUR} + \left[\sum_{j=1}^{n}\left(X_j^T\Sigma^{-1}X_j\right) + kI\right]^{-1}\left[\sum_{a=1}^{N}\sum_{j=1}^{n}\left(X_j^T\Sigma^{-1}y_j^{*(a)}\right) + d\hat{\beta}_{MLE}\right], i = 1.2 \ldots t \quad \textbf{(13)}$$

By taking the derivative of Equations (10) and (11) with respect to Σ respectively and setting the results equal to zero, then we obtain the same result as Equations (7).

Using the IWLS, the RTSUR estimator is defined as Equation (14):

$$\hat{\beta}_{RTSUR}^{(i+1)} = \hat{\beta}_{RTSUR}^{(i)} + \left[\Sigma_{j=1}^{n}X_j^T\Sigma^{-1}X_j + kI\right]^{-1}\Sigma_{j=1}^{n}\left(X_j^T\Sigma^{-1}Y_j^{*}\right) \quad \textbf{(14)}$$

Equation (14) can be simplified as Equation (15):

$$\hat{\beta}_{RTSUR} = [X^T\Sigma^{-1}X + kI]^{-1}X^T\Sigma^{-1}Y^{*} = C_k^{-1}X^T\Sigma^{-1}Y^{*} = W_1Y_j^{*} \quad \textbf{(15)}$$

where $C_k = [X^T\Sigma^{-1}X + kI]$ and $W_1 = C_k^{-1}X^T\Sigma^{-1}$.

The MSE for the $\hat{\beta}_{RTSUR}$ is defined as:

$$MSE[\hat{\beta}_{RTSUR}] = \mathrm{E}\left[(\hat{\beta}_{RTSUR} - \beta)(\hat{\beta}_{RTSUR} - \beta)^T\right] = Cov[\hat{\beta}_{RTSUR}] + Bias[\hat{\beta}_{RTSUR}]Bias[\hat{\beta}_{RTSUR}]^T$$

$$E[\hat{\beta}_{RTSUR}] = C_k^{-1}C\beta$$

$$Cov[\hat{\beta}_{RTSUR}] = C_k^{-1}C\,E(\beta\beta^T)\,C^TC_k^{-1^T} = [C + kI]^{-1}C[C + kI]^{-1^T}$$

$$\mathrm{Bias}\left(\hat{\beta}_{RTSUR}\right) = \left(C_k^{-1}C - I\right)\beta$$
$$= C_k^{-1}(C - C_k)\beta = C_k^{-1}(C - [C + \lambda I])\beta = -\lambda C_k^{-1}\beta = F_1$$

Thus, the MSE for $\hat{\beta}_{RTSUR}$ is Equation (16):

$$MSE[\hat{\beta}_{RTSUR}] = [C + kI]^{-1}C[C + kI]^{-1^T} + k^2[C + kI]^{-1}\beta\beta^T[C + kI]^{-1^T}$$
$$= C_k^{-1}(C + k^2\beta\beta^T)C_k^{-1^T}$$
$$= W_1W_1^T + F_1F_1^T \quad \textbf{(16)}$$

Similarly, using (IWLS), the LTSUR estimator is defined as Equation (17):

$$\hat{\beta}_{LTSUR}^{(i+1)} = \hat{\beta}_{LTSUR}^{(i)} + \left[\sum_{j=1}^{n} X_j^T\Sigma^{-1}X_j + kI\right]^{-1}\left[\sum_{j=1}^{n} X_j^T\Sigma^{-1}X_j + dI\right]\hat{\beta}_{MLE} \quad \textbf{(17)}$$

Equation (17) can be simplified as Equation (18):

Table 6. The SMSE of MLE, RTSUR, and LTSUR under different n, p, ρ_ε and ρ_x with $\varsigma = 40\%$ and m = 4.

Estimators	$n = 100$											
	$p = 5$						$p = 15$					
	$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$			$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$		
	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$
MLE	50.93657	55.9658	61.3657	53.6591	57.9634	65.9657	56.6985	60.0029	68.6952	59.36287	62.9956	70.9638
RTSUR	28.32654	31.6354	33.3639	30.36542	34.6665	37.3268	32.3682	36.3692	40.0009	35.3628	39.3265	43.6392
LTSUR	20.3254	22.3291	24.2525	21.2358	24.3268	25.9991	23.3265	26.9638	28.8935	25.5958	28.8903	33.6355
Estimators	$n = 500$											
	$p = 5$						$p = 15$					
	$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$			$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$		
	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$
MLE	47.7825	51.3262	54.5482	49.3268	53.3699	56.6937	51.5281	55.2677	58.9639	53.3698	56.3263	60.6634
RTSUR	25.5568	27.7811	30.3624	27.0025	29.9637	32.2254	29.9968	31.3142	33.3522	31.1142	34.3621	37.7526
LTSUR	18.2687	20.2224	21.3628	19.0064	22.3268	23.9957	20.9254	24.2587	27.7789	22.3254	25.3692	28.2225

Table 7. The SMSE of MLE, RTSUR, and LTSUR under different n, p, ρ_ε and ρ_x with $\varsigma = 40\%$ and m = 10.

Estimators	$n = 100$											
	$p = 5$						$p = 15$					
	$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$			$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$		
	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$
MLE	53.3325	59.9688	65.5585	55.6668	61.3255	70.2501	58.8576	64.6922	74.3659	61.3911	68.9632	77.9725
RTSUR	31.3655	36.8887	40.0857	32.3642	38.9974	44.3925	35.3646	41.3251	49.9634	39.3258	45.3624	53.6392
LTSUR	23.3628	26.6837	29.3653	24.3624	27.8766	31.3233	25.2834	29.9383	34.4742	27.7787	31.3025	36.968
Estimators	$n = 500$											
	$p = 5$						$p = 15$					
	$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$			$\rho_\varepsilon = 0.2$			$\rho_\varepsilon = 0.9$		
	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$	$\rho_x = 0.85$	$\rho_x = 0.90$	$\rho_x = 0.95$
MLE	50.2568	53.362	56.6937	51.9351	54.6955	59.9636	53.0204	57.9669	64.3626	55.9993	60.0237	69.3059
RTSUR	28.9674	31.3068	35.5862	30.3627	33.3631	39.9632	31.9997	38.8564	40.0023	34.4475	40.0028	44.4626
LTSUR	20.3555	22.7893	25.0025	23.3658	25.9996	29.3625	24.7758	27.3681	317787	25.3622	30.5464	33.8673

$$\hat{\beta}_{LTSUR} = [X^T\Sigma^{-1}X + kI]^{-1}[X^T\Sigma^{-1}X + dI]\hat{\beta}_{MLE}$$
$$= C^{-1}[I + kC^{-1}]^{-1}[C + dI]C^{-1}X^T\Sigma^{-1}y^*$$
$$= C^{-1}[I + kC^{-1}]^{-1}[I + dC^{-1}]X^T\Sigma^{-1}y^*$$
$$= C^{-1}[I + kC^{-1}]^{-1}C^{-1}[C + dI]X^T\Sigma^{-1}y^*$$
$$= C^{-1}[C + kI]^{-1}[C + dI]X^T\Sigma^{-1}y^*$$
$$= C^{-1}C_k^{-1}[C + dI]X^T\Sigma^{-1}y^*$$
$$= W_2y^* \quad \textbf{(18)}$$

where $W_2 = C^{-1}C_k^{-1}[C + dI]X^T\Sigma^{-1}$

The expected value of $\hat{\beta}_{LTSUR}$ is:

$$E[\hat{\beta}_{LTSUR}] = C^{-1}C_k^{-1}[C + dI]C\beta$$

The covariance of $\hat{\beta}_{LTSUR}$ is:

$$Cov[\hat{\beta}_{LTSUR}] = C^{-1}C_k^{-1}[C + dI]C[C + dI]C_k^{-1}C^{-1} = W_2W_2^T$$

The bias of $\hat{\beta}_{LTSUR}$ is:

$$\text{Bias}\,(\hat{\beta}_{LTSUR}) = \left[C^{-1}C_k^{-1}[C + dI]C - I\right]\beta$$
$$= C_k^{-1}[C^{-1}[C + dI]C - C_k^{-1}]\beta$$
$$= C_k^{-1}[C^{-1}[C + dI]C - [C + kI]]\beta$$
$$= -(k - d)C_k^{-1}\beta = F_2$$

The mean square errors of $\hat{\beta}_{LTSUR}$ is defined as Equation (19):

$$MSE[\hat{\beta}_{LTSUR}] = C^{-1}C_k^{-1}[C + dI]C[C + dI]C_k^{-1}C^{-1} + (k - d)^2C_k^{-1}\beta\beta^T C_k^{-1^T}$$
$$= W_2W_2^T + F_2F_2^T \quad \textbf{(19)}$$

when the Tobit SUR model suffers from severe multicollinearity, the RTSUR estimator applies a controlled shrinkage mechanism to stabilize and improve the properties of the parameter estimates. In addition, the LTSUR estimator ensures balances between the bias and variance through its two parameter that achieving more efficient estimates.

4. THE SUPERIORITY OF RTSUR AND LTSUR

To investigate if the RTSUR and LTSUR are outperform the MLE, the MSE will be used as a comparison criterion. Firstly, we use the following lemma:

Lemma 1: Farebrother (1990) [39]. Consider A be an (n × n) positive definite (p.d) matrix and b an (n × 1) vector. Then A - bb^T is p.d if and only if $b^TAb < 1$.

4.1. Comparison between MLE and RSURTE

The MSE expressions in Equations (9) and (16) are used to determine the superiority of the estimators. **Theorem 1:** Consider two linear estimators $\hat{\beta}_{MLE}$ and $\hat{\beta}_{RTSUR}$. If $k > 0$, then $MSE[\beta_{MLE}] - MSE[\hat{\beta}_{RTSUR}]$ is p.d if and only if $\lambda_i\beta^2_i < 1$, I = 1, 2, …, m, where λ_i are the ordered eigenvalues of the $[X^T\Sigma^{-1}X]$ matrix.

Proof:

$$\Delta_1 = MSE[\beta_{MLE}] - MSE[\hat{\beta}_{RTSUR}] = C^{-1} - [W_1W_1^T + F_1F_1^T]$$
$$= C^{-1} - C_k^{-1}(C + k^2\beta\beta^T)C_k^{-1^T}$$
$$= diag\left[\frac{1}{\lambda_i} - \frac{\lambda_i + k^2\beta_i^2}{[\lambda_i + k]^2}\right]$$
$$= diag\left[\frac{[\lambda_i + k]^2 - \lambda_i^2 - \lambda_i k^2\beta_i^2}{\lambda_i[\lambda_i + k]^2}\right]$$
$$= diag\left[\frac{\lambda_i^2 + 2\lambda_i + k^2 - \lambda_i^2 - \lambda_i k^2\beta_i^2}{\lambda_i[\lambda_i + k]^2}\right]$$
$$= diag\left[\frac{2\lambda_i + k^2 - \lambda_i k^2\beta_i^2}{\lambda_i[\lambda_i + k]^2}\right]$$
$$= diag\left[\frac{2\lambda_i + k^2(1 - \lambda_i\beta_i^2)}{\lambda_i[\lambda_i + k]^2}\right]$$

Then Δ_1 is P.d if and only if $\lambda_i\beta^2_i < 1$.

4.2. Comparison between MLE and LTSUR

The MSE expressions in Equations (9) and (16) are used to determine the superiority of the estimators.

Theorem 2: Consider two linear estimators $\hat{\beta}_{MLE}$ and $\hat{\beta}_{LTSUR}$. If > 0, $0 < d < 1$ and $k > d$, then $MSE[\hat{\beta}_{MLE}] - MSE[\hat{\beta}_{LTSUR}]$ is p.d if and only if $\beta^2_i\,(k - d) < 2$.

Proof:

$$\Delta_2 = MSE[\hat{\beta}_{MLE}] - MSE[\hat{\beta}_{LTSUR}] = C^{-1} - [W_2W_2^T + F_2F_2^T]$$
$$\Delta_2 = C^{-1} - C^{-1}C_k^{-1}[C + dI]C[C + dI]C_k^{-1}C^{-1} - (k - d)^2C_k^{-1}\beta\beta^T(C_k^{-1})^T$$
$$= diag\left[\frac{[\lambda_i + k]^2 - [\lambda_i + d]^2 - (k - d)^2\lambda_i\beta_i^2}{\lambda_i[\lambda_i + k]^2}\right]$$
$$= diag\left[\frac{\lambda_i^2 + 2k\lambda_i + k^2 - \lambda_i^2 - 2d\lambda_i - d^2 - k^2\lambda_i\beta_i^2 + 2dk\lambda_i\beta_i^2 - d^2\lambda_i\beta_i^2}{\lambda_i[\lambda_i + k]^2}\right]$$
$$= diag\left[\frac{2k\lambda_i + k^2 + \lambda_i\beta_i^2(2dk - k^2 - d^2) - 2d\lambda_i - d^2}{\lambda_i[\lambda_i + k]^2}\right]$$
$$= diag\left[\frac{2\lambda_i(k - d) - \lambda_i\beta_i^2(k - d)^2 + (k^2 - d^2)}{\lambda_i[\lambda_i + k]^2}\right]$$
$$= diag\left[\frac{\lambda_i(k - d)[2 - \beta_i^2(k - d)] + (k^2 - d^2)}{\lambda_i[\lambda_i + k]^2}\right]$$

For $k > 0$ and $0 < d < 1$, $k > d$ and if, $\beta_i^2(k - d) < 2$ then $\lambda_i(k - d)[2 - \beta_i^2(k - d)] + (k^2 - d^2) > 0$. Thus,

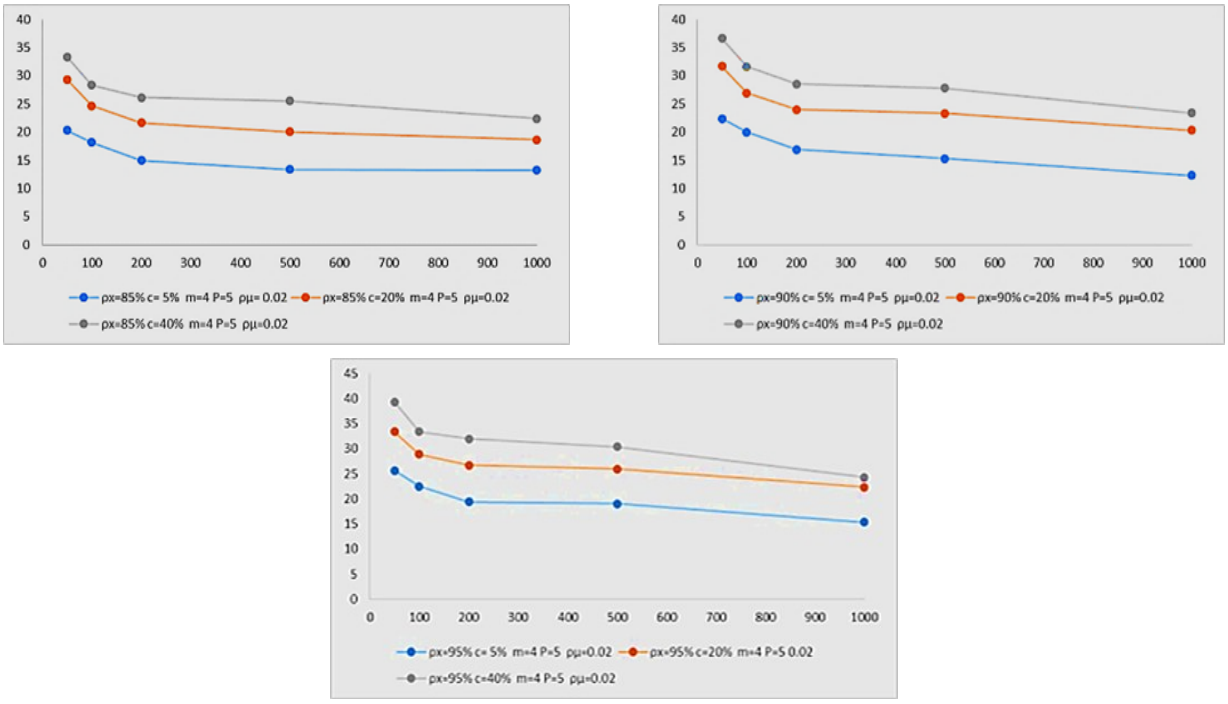

Figure 1. Plots of SMSE values for the RTSUR relative to the number of observations for each equation.

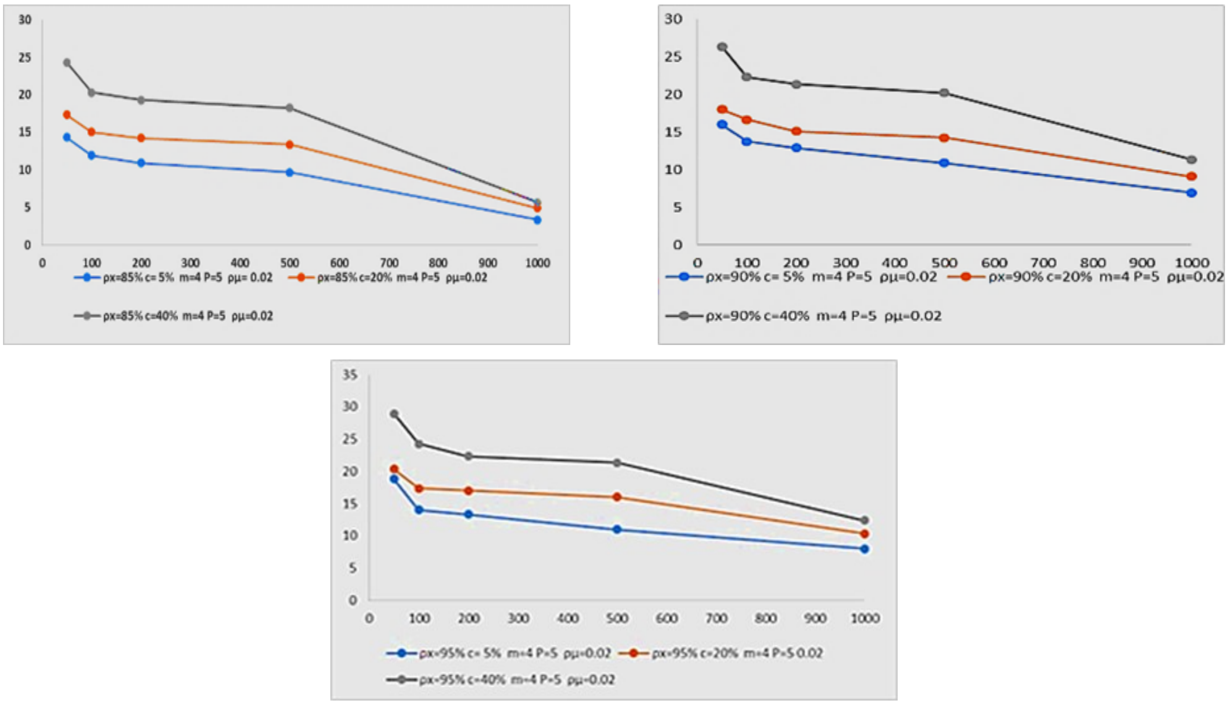

Figure 2. Plots of SMSE values for the LTSUR relative to the number of observations for each equation.

$MSE[\hat{\beta}_{MLE}] - MSE[\hat{\beta}_{LTSUR}]$ is p.d.

4.3. Comparison between RTSUR and LTSUR

The MSE expressions in Equations (9) and (16) are used to determine the superiority of the estimators.

Theorem 3: Consider two linear estimators $\hat{\beta}_{RTSUR}$ and $\hat{\beta}_{LTSUR}$. If $k > 0$ and $0 < d < 1$, $MSE[\hat{\beta}_{RTSUR}] - MSE[\hat{\beta}_{LTSUR}]$ is p.d if $k \geq d/2$ and $W_2 F_W W_2^T \leq 1$. where $F_W = W_1 W_1^T + F_1 F_1^T - F_2 F_2^T$

Proof:

$$\begin{aligned}\Delta_3 = MSE[\hat{\beta}_{RTSUR}] - MSE[\hat{\beta}_{LTSUR}] &= [W_1 W_1^T + F_1 F_1^T] - [W_2 W_2^T + F_2 F_2^T] \\ &= [W_1 W_1^T + F_1 F_1^T - F_2 F_2^T] - [W_2 W_2^T] \\ &= F_W - [W_2 W_2^T]\end{aligned}$$

where $F_W = W_1 W_1^T + F_1 F_1^T - F_2 F_2^T$

In order to clarify that Δ_3 P.d , we proof at the first that F_W is P.d then we apply lemma 1.

$$\begin{aligned}F_W &= C_\lambda^{-1}(C + k^2 \beta\beta^T) C_k^{-1^T} - (k-d)^2 C_k^{-1} \beta\beta^T C_k^{-1^T} \\ &= C_k^{-1}[(C + k^2 \beta\beta^T) - (k-d)^2 \beta\beta^T] C_\lambda^{-1^T} \\ &= C_\lambda^{-1}[C + k^2 \beta\beta^T - k^2 \beta\beta^T + 2dk\beta\beta^T - d^2 \beta\beta^T] C_k^{-1^T} \\ &= C_k^{-1}[C + \beta\beta^T (2dk - d^2)] C_k^{-1^T}\end{aligned}$$

then F_W is P.d if $k \geq d/2$

For lemma 1, Δ_3 is P.d if and only if $W_2 F_W W_2^T <$ 1.

5. THE OPTIMAL VALUE FOR THE PARAMETERS (*k*, *d*)

To determine the d_{opt} for $\hat{\beta}_{LTSUR}$, we fix the k, then take the derivative of $MSE[\hat{\beta}_{LTSUR}]$ in Equation (19) with respect to d and set the result equal to zero.

Let $S = C + dI$ and $\gamma = S\,C^{-1}[C + kI]^{-1} = S\,C^{-1} C_k^{-1}$

$$\begin{aligned}MSE[\hat{\beta}_{LTSUR}] &= C^{-1} C_k^{-1}[C + dI] C [C + dI] C_k^{-1} C^{-1} + (k-d)^2 C_k^{-1} \beta\beta^T C_k^{-1^T} \\ &= C^{-1} C_k^{-1}[C + dI] C [C + dI] C_k^{-1} C^{-1} + C_k^{-1}[C^{-1}[C + dI] C - C_k] \beta\beta^T [C^{-1}[C + dI] C \\ &\quad - C_k] C_k^{-1} \\ &= C\gamma^2 + [\gamma C - I]^2 \beta\beta^T\end{aligned}$$

$$\frac{\partial MSE[\hat{\beta}_{LTSUR}]}{\partial d} = \frac{\partial MSE[\hat{\beta}_{LTSUR}]}{\partial \gamma} \times \frac{\partial \gamma}{\partial d} = [2C\gamma + 2C[\gamma C - I]\beta\beta^T][[C + k]^{-1} C^{-1}] = 0$$

Since $\frac{\partial \gamma}{\partial d} = [C + k]^{-1} C^{-1} \neq 0$, then Equation (20) is obtained.

$$\frac{\partial MSE[\hat{\beta}_{LTSUR}]}{\partial \gamma} = 2C\gamma + 2C[\gamma C - I]\beta\beta^T = 0 \tag{20}$$

Since $\gamma = S\,C^{-1}[C + kI]^{-1}$ then for Equation (20), we find Equation (21):

$$\begin{aligned}2C\gamma + 2C[\gamma C - I]\beta\beta^T &= 2CS\,C^{-1}[C + kI]^{-1} + 2C[S\,C^{-1}[C + kI]^{-1} C - I]\beta\beta^T \\ &= S\,[C + kI]^{-1} + C[S\,[C + kI]^{-1} - I]\beta\beta^T \\ &= S[C + kI]^{-1} + CS\,[C + kI]^{-1}\beta\beta^T - C\beta\beta^T \\ &= S[C + kI]^{-1}(I + C\beta\beta^T) - C\beta\beta^T = 0\end{aligned}$$

and $S_{opt.} = \frac{C\beta\beta^T}{[C+kI][C\beta\beta^T + I]}$,

then $$d_{opt.} = \frac{C\left[[C\beta\beta^T + I]^{-1} \beta\beta^T + [C + kI]\right]}{[C + kI]} \tag{21}$$

To determine the k_{opt} for $[\hat{\beta}_{LTSUR}]$, let $C + kI = G$ and we fix the d, then take the derivative of $MSE[\hat{\beta}_{LTSUR}]$ in Equation (19) with respect to k and set the result equal to zero.

$$\frac{MSE[\hat{\beta}_{LTSUR}]}{\partial k} = \frac{MSE[\hat{\beta}_{LTSUR}]}{\partial \gamma} \times \frac{\partial \gamma}{\partial k} = [2C\gamma + 2C[\gamma C - I]\beta\beta^T][-[C + dI]C^{-1}G^{-2}] = 0$$

Since ,then $\frac{\partial \gamma}{\partial k} = -[C + dI]C^{-1}G^{-2} \neq 0$

$$\frac{MSE[\hat{\beta}_{LTSUR}]}{\partial \gamma} = 2C\gamma + 2C[\gamma C - I]\beta\beta^T = 0 \tag{22}$$

Since $\gamma = S\ C^{-1}\ G^{-1}$, then for Eq. (22), we find

$$\begin{aligned}2C\gamma + 2C[\gamma C - I]\beta\beta^T &= [C + dI]G^{-1}(I + C\beta\beta^T) - C\beta\beta^T = 0 \\ &= [C + dI]G^{-1}(I + C\beta\beta^T) - C\beta\beta^T = 0\end{aligned}$$

and $G_{opt.} = \frac{[C+dI][I + C\beta\beta^T]}{C\beta\beta^T}$

then $$k_{opt.} = \frac{[C+dI][I + C\beta\beta^T] - C^2\beta\beta^T}{C\beta\beta^T} \tag{23}$$

Although k and d are appeared as fixed during differentiation, however, the final values for them are chosen according to the data in each simulation replicate.

6. THE SIMULATION STUDY

In this section, we conduct a simulation study to evaluate the performance of the proposed estimators in comparison to other estimators. The simulation study factors were chosen to correspond with the real-world applications of SUR Tobit regression model. The levels of those factors have been determined so that reflect all practical cases that may be consistent with the SUR model. The multicollinearity levels were selected at low, moderate, and high correlations ($\rho_x = 0.85, 0.90$ and 0.95) since it is compatible with several application for SUR Tobit regression model. To achieve the level of multicollinearity in the SUR model, observations are generated using multivariate normal distribution $MVN_{m(0,\Sigma x)}$, where Σ_x is variance -covariance matrix for the explanatory variables with diagonal elements equal to one and off-diagonal elements equal to (ρ_x). The latent variable was generated according to the following Equation (24):

$$y_{ij}^* = \beta_0 + \sum_{s=1}^{p} \beta_{is} x_{isj} + \varepsilon_{ij},\ i = 1,2,\ldots,m, j = 1,2,\ldots,n \quad \mathbf{(24)}$$

In each simulation replicate, $y_i{*}_j$, $I = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ is obtained by left-censoring at zero, such that $y_{ij} = max\ (0,\ y_i{*}_j)$, so all negative latent values are recorded as zeros. We select the error correlations cross-equation dependence to reflect the disturbances across equations. The random errors are generated using $MVN_{m(0,\ \Sigma\varepsilon)}$, where Σ_ε is the variance-covariance matrix for the equation error with diagonal elements equal one and off-diagonal elements (ρ_ε) representing the correlation between the equations error. We select ρ_ε = 0.20, 0.90. The true coefficients vector is chosen such that $\beta^T\beta$ = 1. The latent variable is censored as follows:

$$y_{ij} = \begin{cases} y_{ij}^*, & if\ y_{ij}^* > 0, \\ 0, & if\ y_{ij}^* \le 0, \end{cases}$$

$i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. When we using multiple censoring levels, that is allows to us study how estimator performance changes when the severity of censoring increases. Three levels of censor were identified, ς = 0.5, 0.20 and 0.40 for all equations. The tuning parameters d and k are selected using use the Equations (21) and (23). The number of equations (m=4,10) was selected to include both simple and complex multivariate systems. In addition, the number of observations for each equation (n) was select at different levels to represent small and moderately samples which used in many empirical studies. The values of were chosen to represent small to moderately large regression systems. The number of observations per equation is set at two levels (n=100 and 500). The p was select as (p= 5 and 15) to evaluate the behavior for penalized estimators (RTSUR, LTSUR) when the dimensionality for the model grows, especially when sample sizes are fixed. We run the simulation r=1500 times and we compared between the estimators in Equations (8), (15) and (18) according to the selected factors, using the simulated mean square error (SMSE) criteria, which defined as Equation (25):

$$SMSE(\hat{\beta}) = \frac{\sum_{r=1}^{1500}(\hat{\beta}_{sim.} - \beta)^T(\hat{\beta}_{sim.} - \beta)}{1500} \quad \mathbf{(25)}$$

where $\hat{\beta}_{sim.}$ is the estimator in the r^{th} simulation. The factors for the simulation study are summarized in Table 1.

The simulation follows several steps. First, the data are generated through determining the values of m, n, and r then, the explanatory variables are generated for each equation and the true regression coefficients and the error covariance matrix Σ_ε are specified. According to the data in first step, the latent dependent variables are generated. For the

Table 8. Describe the raw data using in the study.

Variables	Descriptions- Unit
Dependent Variable	
Carbon monoxide (CO)	The total (CO) concentrations during the year - (µg/m³)
Sulfur dioxide (SO_2)	The total (SO_2) concentrations during the year - (µg /m³)
Nitrogen dioxide (NO_2)	The average of (NO_2) concentrations during the year- (µg /m³)
Independent Variables	
Humidity (RH)	Annual average humidity - (%)
Wind speed(WS)	Annual average wind speed - (km/h)
Atmospheric pressure(AP)	Annual average atmospheric pressure (inches of mercury - (inHg))
Temperature(TM)	Annual average temperature - (Fahrenheit)
Precipitation(PR)	Annual average precipitation - (%)

second step, the censoring rule to obtain the observed outcomes is apply according to:

$$y_{ij} = \begin{cases} y_{ij}^{*}, & if\ y_{ij}^{*} > 0 \\ 0, & if\ y_{ij}^{*} \leq 0 \end{cases}, i = 1,2, \dots, m, j = 1,2, \dots, n$$

For the next step, the SUR Tobit regression model is formulated, yielding the log-likelihood, which is fundamental for the implementation of the MCECM algorithm. The MCECM algorithm is then applied to iteratively update the model parameters and the latent variables until convergence is achieved. Following this, the standardized mean squared error (SMSE) is calculated using Equations (25). Finally, the simulation results are summarized in Tables 2 – 7, which present a comparison among the MLE, RTSUR, and LTSUR estimators under several experimental conditions, including different sample sizes, numbers of equations, numbers of parameters, levels of multicollinearity, error correlations, and censoring rates.

Figures 1 and 2 illustrate the behavior of the RTSUR and LTSUR estimators at different levels of multicollinearity and censoring. From Figures 1 and 2, we observe that the RTSUR and LTSUR estimators perform better as the number of observations for each equation increases. Additionally, regarding the LTSUR estimator, the decrease in SMSE slows as the value of n increases, especially when ς increases. At the same time, the SMSE values converge as n and ς increase.

For Table 2, the SMSE for the estimators increase when multicollinearity (ρ_x) or error correlation (ρ_ε) increases. in addition, the Liu-type SUR (LTSUR) has the smallest SMSE and the ridge SUR (RTSUR) is the second best, and the MLE has the largest SMSE. For example, when n = 100, p = 5, ρ_ε = 0.2 and ρ_x = 0.85, SMSE (MLE) = 33.61, SMSE(RTSUR) = 18.15 and SMSE (LTSUR) = 11.21 then the LTSUR reduces SMSE by 66.6% relative to MLE and RTSUR reduces SMSE by 46.0% relative to MLE.

For Table 3, when the system has a larger dimension m = 10, the SMSEs are generally higher than when m = 4. The LTSUR gives the smallest SMSE, RTSUR has the second smallest, and MLE has the largest. For example, with n = 100, p = 5, ρ_ε = 0.2 and ρ_x = 0.85, the SMSE (MLE) = 39.90, MSE(RTSUR) = 19.89 and SMSE(LTSUR) = 12.25 then the LTSUR reduces SMSE roughly by 69% compared with MLE. In addition, m increases the SMSE but preserves the ordering of estimators.

For Table 4, when censoring severity increases to 20%, SMSEs are increase for all estimators and the advantage of penalized estimators are persisting for all factors. Example: for n = 100, p = 5, ρ_ε = 0.2, ρ_x = 0.85, SMSE(MLE) 15.03. Comparing when ς=5% in Table 2 whith ς=20% in Table 4 at the same levels for factors shows 36.7% increase in SMSE,s (33.61 to 45.94), indicating that censoring substantially degrades estimation accuracy. When the censoring proportion increase, the variance and bias of all estimators increase but the penalized estimators relieve some of the impact but do not fully compensation the loss of information caused by heavy censoring.

For Table 5, when m = 10 and ς = 20%, we find that, the SMSE (LTSUR) less than SMSE (RTSUR) less than SMSE(MLE). In addition, the gap between MLE and shrinkage estimators often widens with both higher ς and m, illustrating those penalization estimators have return especially in difficult systems.

For Table 6, when we have extremely heavy censoring (ς = 40%) the SMSEs grow further and the relative improvements of shrinkage methods remain notable. Example: When n = 100, p = 5, ρ_ε = 0.2, ρ_x = 0.85, we find that SMSE(MLE)=50.94, SMSE(RTSUR)=28.33, SMSE(LTSUR)=20.33. Increasing ν from 20% to 40% lead to increase SMSE, s by about 10.9% (45.94 to 50.94).

For Table 7, at the largest and most difficult systems such that m, ς, ρ_x are high, the SMSE for the penalized estimators tends to increase.

By analyzing the results of the simulation study, we find that, for all levels of censoring, the MLE is the worst estimator, while the LTSUR is the best estimator compared to the others across all factors. The RTSUR estimator performs well at different levels of correlation between explanatory variables, but the LTSUR estimator is superior when dealing with high correlations between explanatory variables. Additionally, for all estimators, as the number of variables per equation, the number of equations, the correlation between equation errors, and the correlation between explanatory variables increase, the SMSE values also increase. Conversely, when the number of observations and levels of censoring increase, the SMSE values

Table 9. The estimated coefficients and MSE values.

Estimators	Coefficients						MSE
	β_{10}	β_{11}	β_{12}	β_{13}	β_{14}	β_{15}	
	1.234	-0.247	-0.362	0.524	-0.437	-0.602	
	β_{20}	β_{21}	β_{22}	β_{23}	β_{24}	β_{25}	
LTSUR	1.511	-0.336	-0.197	0.384	-0.603	-0.443	9.367
	β_{30}	β_{31}	β_{32}	β_{33}	β_{34}	β_{35}	
	0.935	-0.509	-0.491	0.118	-0.934	-0.698	
	β_{10}	β_{11}	β_{12}	β_{13}	β_{14}	β_{15}	
	1.391	-0.328	-0.304	0.704	-0.559	-0.715	
	β_{20}	β_{21}	β_{22}	β_{23}	β_{24}	β_{25}	
RTSUR	1.536	-0.401	-0.224	0.311	-0.712	-0.491	11.024
	β_{30}	β_{31}	β_{32}	β_{33}	β_{34}	β_{35}	
	1.001	-0.614	-0.581	0.196	-1.051	-0.787	
	β_{10}	β_{11}	β_{12}	β_{13}	β_{14}	β_{15}	
	2.218	-0.589	-0.692	1.024	-0.744	-0.827	
	β_{20}	β_{21}	β_{22}	β_{23}	β_{24}	β_{25}	
MLE	1.914	-0.905	-0.596	0.829	-1.124	-0.903	21.354
	β_{30}	β_{31}	β_{32}	β_{33}	β_{34}	β_{35}	
	1.334	-1.031	-0.746	0.326	-1.423	-1.203	

decrease. Both RTSUR and LTSUR demonstrate a strong ability to handle high levels of censoring.

6.1. Application to Environment Data

In this section, the performance of the proposed estimator will be demonstrated by application to real data. We used annual average data on air pollution and various weather factors in the city of Cairo between 1990 and 2020. The weather-related data (relative humidity (RH), wind speed (WS), atmospheric pressure (AP), temperature (TM), and precipitation (PR)) were obtained from the Al-Nozha Weather Station, available at (https://www.wunderground.com/history/monthly/eg/al-nozha/HECA). The available climate data is monthly, so the average data for the months of the year was estimated to obtain the annual average. The air pollution data (annual concentrations of air pollutants, which include of carbon monoxide (CO), sulfur dioxide (SO_2), and nitrogen dioxide (NO_2)) were obtained through satellite reanalysis (MERRA -2) and are available at (https://gmao.gsfc.nasa.gov/reanalysis/MERRA-2/data_access/). This date were collected in the May, 2025. The main components of air pollutants considered are three elements: carbon monoxide (CO), sulfur dioxide (SO_2) and nitrogen dioxide (NO_2). Liu et al. (2020) noted that there is direct relationship between air pollution levels and meteorological conditions such as relative humidity (RH), wind speed (WS), atmospheric pressure (AP), temperature (TM) and precipitation (PR) [40]. In fact, the concentrations of most air pollutants are influenced by weather conditions, though the level of influence varies depending on the type of pollutant. Liu et al, (2020) also demonstrated that, there is a significant negative correlation between air pollutants and wind speed, rainfall, and relative humidity, and a positive correlation with atmospheric pressure [38]. We use the levels of air pollutants as the dependent variable crossing of four main gas elements (CO, SO_2, NO_2). With four blocks, we have m=4 equations in our SUR model with 30 observations per equation. The RH, WS, AP, TM and (PR) variables are use as independent variables. To align the pollution data with the meteorological records, all pollutant series were converted to annual averages. The Egyptian Ministry of Environment, according to Environmental Law No. 4 ,1994 and its executive regulations, has determined the permissible levels of air pollutants (CO = 30 µg/m³/ year, SO_2 =6 0 µg/m³/ year and NO_2 = 40 µg/m³/ year). These levels were using as a censoring thresholds for the dependent variables. Information about the raw data is summarized in Table 8.

The Tobit SUR mode is specified as:

$$CO_t^{*} = \beta_{10} + \beta_{11}RH_t + \beta_{12}WS_t + \beta_{13}AP_t + \beta_{14}TM_t + \beta_{15}PR_t + \varepsilon_{1t}$$
$$NO_{2t}^{*} = \beta_{20} + \beta_{21}RH_t + \beta_{22}WS_t + \beta_{23}AP_t + \beta_{24}TM_t + \beta_{25}PR_t + \varepsilon_{2t}$$
$$SO_{2t}^{*} = \beta_{30} + \beta_{31}RH_t + \beta_{32}WS_t + \beta_{33}AP_t + \beta_{34}TM_t + \beta_{35}PR + \varepsilon_{3t}$$

where CO, SO_2 and NO_2 are determined as follows:

$$CO_t = \begin{cases} CO_t^{*}, & if\ CO_t^{*} > 30, \\ 0, & if CO_t^{*} \leq 30, \end{cases}$$

$$NO_{2t} = \begin{cases} NO_{2t}^{*}, & if\ NO_{2t}^{*} > 40, \\ 0, & if\ NO_{2t}^{*} \leq 40, \end{cases}$$

$$SO_{2t} = \begin{cases} SO_{2t}^{*}, & if\ SO_{2t}^{*} > 60, \\ 0, & if SO_{2t}^{*} \leq 60, \end{cases}$$

The tuning parameters d and k are selected using use the Equations (21) and (23). We used the condition number $CN = \sqrt{\lambda_{max}/\lambda_{min}}$ to assess the level of multicollinearity. To determine the value of CN, we obtained the eigenvalues of the matrix $X^T\Sigma^{-1}X$, which were 4016.901, 399.563, 33.719 and 2.048. We found that CN = 44.287, indicating a high level of multicollinearity. We used β_{ij} = 1 for i=1,2,3 and j=0,1,2,…,5 as the true values for the coefficients. The MSE depend on correlation between the independent variables and penalty parameters rather than on the specific value of β. Hence, choosing β_{ij} = 1 does not affect the superiority of estimators.

In Table 9, we summarize the results of comparison between the RTSUR, LTSUR and MLE estimators according to MSE criteria. These results show that the RTSUR and LTSUR estimators perform better at high level of multicollinearity, while MLE is the worst estimator. The signs for the coefficients are consistent across all estimators and align with climate-related studies. For all gas elements (CO, SO_2, NO_2), the coefficients for the RH, WS, TM and PR variables are negative while the coefficient for the (AP) variable is positive. According to the results of climate pollution data, we note that it is consistent with the theoretical expectations. For Figure 3, the LTSUR and RTSUR estimators are shrink the estimated coefficients toward zero which explain the penalization effect. The shrinkage is stronger in LTSUR than in RTSUR, which explains why the MSE for RTSUR is slightly higher than LTSUR. For all coefficients in the SUR regression model, we nots that, the MLE coefficients tend to be more extreme, while the penalized estimators seem more moderate. So that, it is produce more stable and consistent estimates. The LTSUR estimator has the smallest MSE among all estimators, indicating its superior performance, while the MLE estimator produced the largest MSE, indicating the poorest performance. According to the results, the proposed LTSUR estimator has more stable and reliable for estimates the relationships between meteorological factors and air pollution. This allows decision maker to better identify which weather variables are effect of the pollutant concentrations in Cairo.

7. INTERPRETATION OF THEORETICAL RESULTS

The theoretical results presented in the previous section present that, the MSE of the ridge and Liu-type estimators are less MSE for MLE estimator under many conditions. This section provides an explanation of these conditions and its practical results for applied researchers.

The condition for superiority the Ridge estimator over MSE for MLE is $\lambda_i\beta^2_i < 1$. Since λ_i play a critical role. These eigenvalues measure the amount of information available in each principal component direction of the covariate space. When λ_i is small, this is mostly due to multicollinearity. Then the condition $\lambda_i\beta^2_i < 1$ in practically means that the ridge estimator is one of the best estimators when the penalty is large enough to reduce the inflated variance produced result of multicollinearity, but not large to suppresses the true underlying effect. This condition explains why ridge estimator has lower MSE. The penalized estimators, such as ridge and Liu-Type shrinking coefficients toward zero which lead to stabilizes estimates and reduces variance. So that MSE becomes smaller than that of the MLE.

The theoretical results closely with simulation study in the behavior of biased estimators in high-dimensional such that, high values for n, m, ς, and p or ill-conditioned environments such that high levels for ρ_x. Therefore, the results which obtained for the penalized estimators for SUR Tobit model are consistent with the statistical literature, where it is achieve significant improvements for the accuracy for the estimator whenever the model suffers from multicollinearity.

8. CONCLUSIONS

In this paper, we introduced two penalized estimators for the Tobit SUR model (SURT) to address multicollinearity. These estimators are the Ridge Tobit SUR (RTSUR) and Liu-Type Tobit SUR (LTSUR). The study achieved its main objectives of developing estimators that improve estimation efficiency under multicollinearity and evaluating their performance both theoretically and empirically. For the theoretical analysis, we using the MSE criterion that show that the LTSUR estimator provides superior performance compared to RTSUR and MLE estimators. A simulation study was conducted to demonstrate the performance of the estimators under various conditions, using SMSE as the criterion for comparison. The factors considered in this comparison included censor levels, multicollinearity levels, correlation between the equation errors, sample size, number of equations, and number of parameters. The results of the simulation indicate that the RTSUR and LTSUR estimators perform well at high levels of multicollinearity and high levels of censoring. To examine the behavior of the estimators with real data, we applied them to air pollution data from Cairo, Egypt, which confirmed the superiority of the proposed estimators. This study shown that the penalized estimators, especially the Liu-Type estimator, are highly effective tools for improving estimation in SUR Tobit regression models, which suffer from multicollinearity. Moreover, the proposed estimator can be applied in other multivariate censored regression contexts, since it has both theoretical and practical advantages over classical estimators. The results for the application of air pollution data for Cairo support better decision making in environmental policy, economic and social sciences. The current study was applied to environmental data for the city of Cairo, which is within the study's limitations. In addition, this study assumed that the error term has normal distribution for each equation in the TSUR regression model.

INDEX

$\boldsymbol{m}$	Number of equations in the SURT regression model
$\boldsymbol{n}$	Number of observations for each equation
$\boldsymbol{p}$	Number of explanatory variables
ς	Censoring levels
$\boldsymbol{\rho_x}$	Multicollinearity levels
$\boldsymbol{\rho_\varepsilon}$	Correlation between the equation errors
k* and *d	Tuning parameters for Liu-Type estimator
$\boldsymbol{S^h}$	Censoring regime vector indicating pattern of censored (0) and uncensored (+) values
$\boldsymbol{h}$	Index for censoring regime vectors S^h, where $h = 1, 2, \ldots, 2^m$
$\boldsymbol{r}$	Number of censored components (zeros) in a given regime vector S^h
$\boldsymbol{a}$	Index for Monte Carlo replications in the MCECM algorithm
N	Number of samples

Figure 3. The coefficient estimates across LTSUR, RTSUR and MLE estimators.

y_{ij}^{*}	Latent (unobserved) dependent variable for equation i and observation j
y_{ij}	Observed dependent variable
x_{ij}	Row vector of explanatory variables for equation i and observation j
β_i	$(p + 1)$ x 1 coefficient vector for equation I
β	Stacked coefficient vector. $\beta = (\beta_1^T, \beta_2^T, \ldots, \beta_m^T)^T$
ε_j	Error vector $\varepsilon_j = (\varepsilon_{1j}, \varepsilon_{2j}, \ldots, \varepsilon_{mj})^T \sim N_m(0, \Sigma)$
Σ	m x m symmetric positive definite covariance matrix of errors
λ_i	Eigenvalues of the $[X^T \Sigma^{-1} X]$ matrix, $i = 1, 2, \ldots, m$
θ	Parameter vector $\theta = (\beta, \Sigma)$
$L_j^{(Sh)}$	Likelihood contribution of the j^{-th} observation given censoring regime S^h
$\hat{\beta}_{RTSUR}$	Ridge estimator for seemingly unrelated regression Tobit model
$\hat{\beta}_{LTSUR}$	Liu-Type estimator for seemingly unrelated regression Tobit model

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The author wrote the paper and made the discussion properly.

Conflicts of Interest

The authors declare no conflict of interest.

DECLARATION OF GENERATIVE AI

Not applicable.

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