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Research Article

Truncated Transmuted Exponential Distribution with Different Estimation Methods and Applications

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Abstract
In this study, we study and discuss a new truncated flexible distribution named the truncated transmuted exponential distribution. The probability density function of the new suggested model is decreasing, but its hazard rate function is increasing. Some fundamental statistical properties of the new distribution are computed such as moments, incomplete moments, moment generating function, quantile function, mean residual lifetime, mean past lifetime, order statistics, and extropy. Various approaches of estimation are considered include maximum likelihood, least squares, weighted least squares, Cramér-Von-Mises, maximum product of spacings, Anderson-Darling, right-tail Anderson-Darling and percentile methods. A simulation study is established to assess the accuracy of estimates through some measures. The importance of the truncated transmuted exponential model is demonstrated using real lifetime data, and its goodness-of-fit is evaluated against alternative models. This study offers a more accurate match to the data compared to other competing models.

Keywords: transmuted exponential distribution, quantile function, maximum likelihood method, simulation

1. INTRODUCTION

Developing and analyzing lifetime data are crucial in various practical domains including as health, engineering, and economics. Data have been analyzed using different lifetime distributions. The chosen probability model or distribution greatly influences the efficacy of the methods employed in a statistical analysis. As a result, significant effort has been dedicated to developing extensive categories of traditional probability distributions together with relevant statistical methods. There are several important difficulties when the actual data does not align with established or conventional probability models [1].

A well-established technique for creating new families of distributions with greater flexibility is to add parameters to an existing distribution. Shaw and Buckley were the first to propose adding a new parameter to an existing distribution to provide more distributional flexibility [1]. This approach is intriguing. They also employed the quadratic rank transmutation map, or QRTM, to construct a flexible family of distributions. The generated family, also known as the transmuted extended distribution, integrates the parent distribution as a specific instance and offers additional flexibility in representing various types of data.

The family of transmuted distributions has been widely studied in recent years. Shaw and Buckley introduced the concept of the transmuted distribution, which is a pretty simple idea [1]. Transmuted distributions have garnered significant interest in the field of statistics since then, leading to the development of various transmuted-type distributions by different authors, derived from established models. For instance, the transmuted Rayleigh distribution considered by Merovci [2], the transmuted modified Weibull model discussed by Khan and King [3], the transmuted inverse Weibull distribution presented by Khan et al. [4], and the transmuted generalized log logistic distribution with four parameters investigated by Adeyinka et al. [5]. On the other hand, the transmuted Pareto distribution was discussed by Merovci and Puka [6], a new cubic transmuted power-function distribution was submitted by Ahsan-ul-Haq et al. [7], the performance of transmuted logistic distribution was proposed by Samuel [8], a transmuted modified power-generated family of distributions with practice on sub models in insurance and reliability was introduced by Naz et al. [9] transmuted Gumbel type-II distribution with applications in diverse fields of science by Ahmad et al. [10]. Aryal et al. have researched the

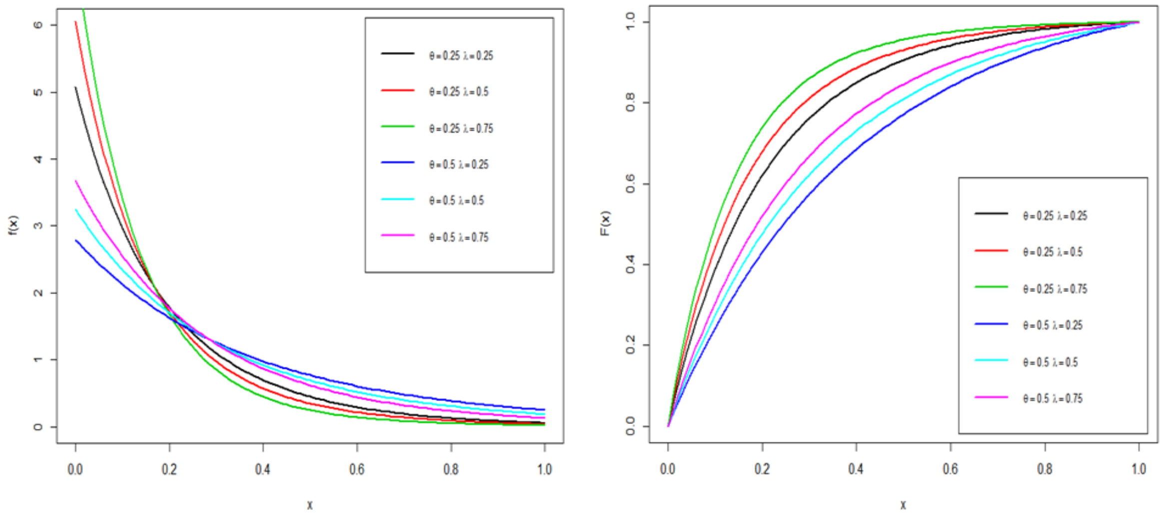

Figure 1. Graphs of the CDF and PDF for the TTE model with varying parameters values.

transmuted Gumbel distribution and found that it can be used to represent climate data [11]. Aryal and Tsokos [12] developed the transmuted Weibull model by applying the quadratic rank transmutation map on the Weibull distribution [12]. They examined the mathematical properties of this distribution and calculated the maximum likelihood estimates of the parameters. When applied to real data sets, it was shown that the transformed distributions exhibited greater flexibility compared to the original distributions. Eldessouky et al. introduced the transmuted power unit inverse Lindley distribution [13]. Badr et al. discussed the transmuted odd Fréchet-G family of distributions [14]. Elgarhy et al. proposed the transmuted Kumaraswamy quasi-Lindley and transmuted Kumaraswamy Lindley distributions [15][16].

A random variable X is said to have a transmuted exponential (TE) distribution if its probability density function (PDF) and cumulative distribution function (CDF) are respectively given by Equations 1 and 2;

$$f(x;\theta,\lambda) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \left[1 - \lambda + 2\lambda e^{-\frac{x}{\theta}}\right], \quad x > 0, \theta > 0, |\lambda| \leq 1, \quad \textbf{(1)}$$

$$F(x;\theta,\lambda) = \left[1 - e^{-\frac{x}{\theta}}\right]\left[1 + \lambda e^{-\frac{x}{\theta}}\right], \quad x > 0, \theta > 0, |\lambda| \leq 1, \quad \textbf{(2)}$$

where, θ is the scale parameter and λ is the ransmuted parameter.

Truncated distributions are commonly used in various scientific disciplines such as communication networks, economics, hydrology, material sciences, and physics. A truncated distribution occurs when the original distribution is limited to a smaller range of values. Truncated data is a well-recognized issue in the field of reliability, particularly when assessing failure rates of objects. Truncation refers to the limitation on obtaining information about items that are outside a specific scope. Truncation in manufacturing involves selecting a smaller set of goods from a broader population after eliminating those that did not fit the requirements [17].

The shortened versions of typical statistical distributions are suggested to address the analysis of truncated data in several domains. Singh et al. examined the truncated Lindley distribution [17]. Ahmed et al. explored the application of the truncated Birnbaum-Saunders (BS) distribution to improve an actuarial forecasting model, particularly for analyzing data related to insurance payments with deductibles [18]. Aban et al. [19] and Zaninetti et al. [20] examined how the truncated Pareto distribution can be used for statistical analysis of star masses and asteroid sizes. Various uses of the truncated Weibull distribution have been identified, such as analyzing tree diameter data at a specific threshold level, modeling forest diameter distribution, characterizing recorded Portuguese fire size distribution, and analyzing seismological data. Readers seeking more information on the truncated Weibull distribution and associated references can refer to Murthy et al. [21] and the new work by Zhang et al. [22] which focuses on the truncated Weibull distribution. Lately, Elgazar et al. discussed the truncated version of the moment exponential distribution with its properties and application [23]. Hussein and Ahmed suggested the idea of the truncated Gompertz-exponential on the interval [0,1] with specific characteristics and practical uses [24]. Almarashi et al. discussed a novel truncated Muth produced family of distributions [25]. A new zero-truncated distribution and its application to count data is discussed by Elah et al. [26]. Ahmed et al. proposed the truncated inverse power Ailamujia distribution with its properties and applications [27]. Semary et al. discussed the univariate and bivariate extensions of the truncated inverted arctan power distribution with applications [28]. Elgarhy et al. studied bayesian and non-bayesian estimations of truncated inverse power Lindley distribution under progressively type-II censored data with applications [29]. Elgarhy et al. introduced bayesian inference using MCMC algorithm of sine truncated Lomax distribution with application [30]. Alotaibi et al. proposed the truncated Cauchy power Weibull -G class of distributions with bayesian and non-bayesian inference modelling for COVID-19 and carbon fiber data [31]. Soliman et al. discussed the statistical properties and applications of a new truncated Zubair- generalized family of distributions [32].

This work aims to present a novel truncated distribution called the right truncated transmuted exponential (TTE) distribution. We determine basic characteristics of the new model and investigate several techniques for estimation. This article seeks to accomplish the following objectives construct a new flexible distribution that can display decreasing data. Its hazard rate function (HRF) can be increasing, to establish and calculate some of the key statistical properties, including moments, incomplete moments, moment generating function, quantile function, mean residual lifetime, mean past life time and order statistics, and to obtain the point estimators of the unknown parameters of the TTE model using the maximum likelihood (ML), least square (LS), weighted LS (WLS), maximum product of spacings (MPS), Cramér-Von-Mises (CM), Anderson-Darling (AD), right-tail Anderson-Darling (RAD) and percentile (P) estimation methods. Through the implementation of the Monte Carlo simulation approach, the accuracy of the proposed model estimates is evaluated; to examine the practical significance of the model suggested, three different real-world datasets are used to compare it to well-known alternatives.

2. TRUNCATED TRANSMUTED EXPONENTIAL DISTRIBUTION

The PDF of a random variable X, which follows the right truncated transmuted exponential distribution can be expressed as Eq. 3.

$$f_{TTE}(x,\theta,\lambda)=\frac{f(x)}{\int_0^1 f(x)dx}=\frac{A}{\theta}e^{-\frac{x}{\theta}}\left[1-\lambda+2\lambda e^{-\frac{x}{\theta}}\right],$$

$$0<x<1, \theta>0, |\lambda|\le 1, \qquad (3)$$

where, $A=\frac{1}{\left[1-e^{-\frac{1}{\theta}}\right]\left[1+\lambda e^{-\frac{1}{\theta}}\right]}$.

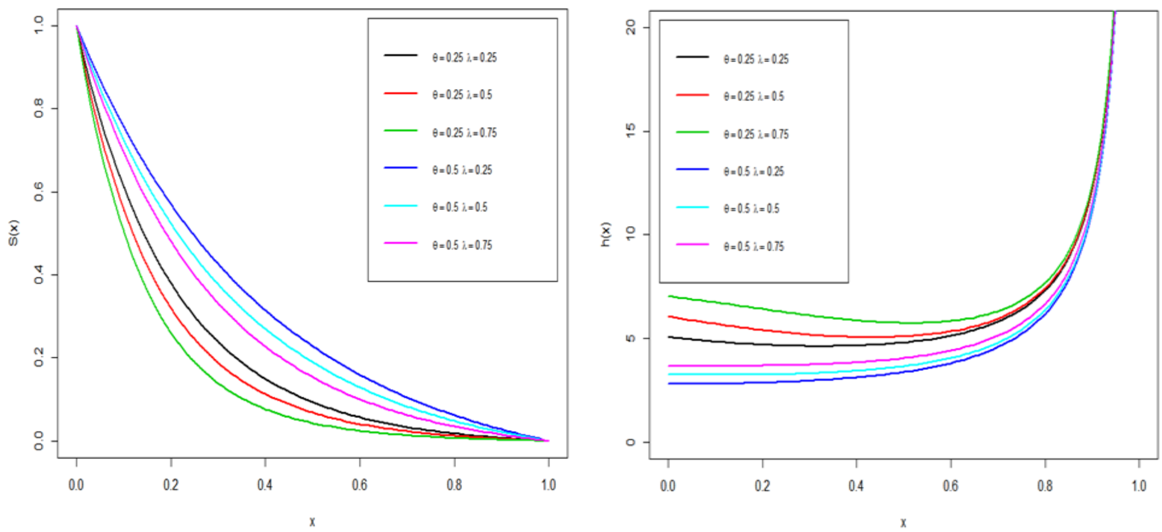

Figure 2. The graphs of reliability and HRF of the TTE distribution for several values of parameters.

The CDF related to (3) is given by:

$$F_{TTE}(x,\theta,\lambda) = \frac{F(x,\theta,\lambda) - F(0,\theta,\lambda)}{F(1,\theta,\lambda) - F(0,\theta,\lambda)} = A\left[1 - e^{-\frac{x}{\theta}}\right]\left[1 + \lambda e^{-\frac{x}{\theta}}\right],$$

$$0 < x < 1, \theta > 0, |\lambda| \leq 1. \quad \textbf{(4)}$$

Figure 1 displays the graphs of the PDF and CDF for the TTE distribution with various values of θ and λ. The PDF of the TTE model may exhibit a right-skewed distribution.

The reliability function and HRF of the TTE distribution can be expressed by using PDF (3) and CDF (4) respectively as follows:

$$S_{TTE}(x,\theta,\lambda) = 1 - F_{TTE}(x,\theta,\lambda) = 1 - A\left[1 - e^{-\frac{x}{\theta}}\right]\left[1 + \lambda e^{-\frac{x}{\theta}}\right],$$

and

$$h_{TTE}(x,\theta,\lambda) = \frac{f_{TTE}(x,\theta,\lambda)}{S_{TTE}(x,\theta,\lambda)} = \frac{\frac{1}{\theta}e^{-\frac{x}{\theta}}\left[1 - \lambda + 2\lambda e^{-\frac{x}{\theta}}\right]}{\left[1 - e^{-\frac{1}{\theta}}\right]\left[1 + \lambda e^{-\frac{1}{\theta}}\right] - \left[1 - e^{-\frac{x}{\theta}}\right]\left[1 + \lambda e^{-\frac{x}{\theta}}\right]}.$$

Figure 2 illustrates the reliability and hazard rate functions over various values of θ and λ, showing their characteristics. The curve of the HRF of the TTE model can be either increasing or J-shaped.

Also, the reserved HRF and cumulative HRF can be determined respectively as follows:

$$\tau_{TTE}(x;\theta,\lambda) = \frac{f_{TTE}(x;\theta,\lambda)}{F_{TTE}(x;\theta,\lambda)} = \frac{\frac{1}{\theta}e^{-\frac{x}{\theta}}\left[1 - \lambda + 2\lambda e^{-\frac{x}{\theta}}\right]}{\left[1 - e^{-\frac{x}{\theta}}\right]\left[1 + \lambda e^{-\frac{x}{\theta}}\right]},$$

and

$$H_{TTE}(x;\theta,\lambda) = -\log\left(S_{TTE}(x;\theta,\lambda)\right) = -\log\left(1 - A\left[1 - e^{-\frac{x}{\theta}}\right]\left[1 + \lambda e^{-\frac{x}{\theta}}\right]\right).$$

3. BASIC PROPERTIES

This section presents certain statistical features of the TTE distribution. These properties are studied respectively, such as the moments, moment generating and characteristic functions, incomplete moments, Lorenz and Bonferroni curves, quantile function, mean residual lifetime, mean past lifetime, order statistics and extropy.

3.1. Moments and Associated Measures

As the moments are fundamental and significant in any statistical study. So, we calculate the r^{th} moment of the truncated transmuted exponential distribution. Let X has the PDF (3), then $\acute{\mu}_r$ is determined as Equation (5).

$$\begin{aligned}\acute{\mu}_r &= \int_0^1 x^r f_{TTE}(x;\theta,\lambda)dx \\ &= \frac{A}{\theta}\int_0^1 x^r e^{-\frac{x}{\theta}}\left[1 - \lambda + 2\lambda e^{-\frac{x}{\theta}}\right]dx, \\ &= A\theta^r\left[(1-\lambda)\left[\Gamma(r+1) - \Gamma\left(r+1,\frac{1}{\theta}\right)\right] \right. \\ &\quad \left. + \lambda 2^{-r}\left[\Gamma(r+1) - \Gamma\left(r+1,\frac{2}{\theta}\right)\right]\right].\end{aligned} \quad \textbf{(5)}$$

The mean and variance of TTE distribution can be determined as:

$$\mu = E(X) = A\theta\left[(1-\lambda)\left[1 - \Gamma\left(2,\frac{1}{\theta}\right)\right] + \frac{\lambda}{2}\left[1 - \Gamma\left(2,\frac{2}{\theta}\right)\right]\right],$$

and

$$\begin{aligned}\sigma^2 = V(X) &= E(X^2) - (E(X))^2, \\ &= A\theta^2\left[(1-\lambda)\left[2 - \Gamma\left(3,\frac{1}{\theta}\right)\right] + \frac{\lambda}{4}\left[2 - \Gamma\left(3,\frac{2}{\theta}\right)\right]\right] \\ &\quad - \left[A\theta\left[(1-\lambda)\left[1 - \Gamma\left(2,\frac{1}{\theta}\right)\right] + \frac{\lambda}{2}\left[1 - \Gamma\left(2,\frac{2}{\theta}\right)\right]\right]\right]^2.\end{aligned}$$

By substituting in equation (5), we can determine $E(X^2)$, $E(X^3)$ and $E(X^4)$ as follows:

$$E(X^2) = A\theta^2\left[(1-\lambda)\left[2 - \Gamma\left(3,\frac{1}{\theta}\right)\right] + \frac{\lambda}{4}\left[2 - \Gamma\left(3,\frac{2}{\theta}\right)\right]\right],$$

$$E(X^3) = A\theta^3\left[(1-\lambda)\left[6 - \Gamma\left(4,\frac{1}{\theta}\right)\right] + \frac{\lambda}{8}\left[6 - \Gamma\left(4,\frac{2}{\theta}\right)\right]\right],$$

and

$$E(X^4) = A\theta^4\left[(1-\lambda)\left[24 - \Gamma\left(5,\frac{1}{\theta}\right)\right] + \frac{\lambda}{16}\left[24 - \Gamma\left(5,\frac{2}{\theta}\right)\right]\right].$$

The skewness and kurtosis of the TTE distribution can be easily confirmed by utilizing the specified relationship:

$$S = \frac{E(X^3) - 3E(X)E(X^2) + 2(E(X))^3}{(V(X))^{3/2}},$$

and

Table 1. Mean, MSE, and Rbias for several estimation techniques of TTE model at different sample θ = 0.2 and λ = 0.25.

n	Est.	Est. ; Par.	MLE	MPS	LS	WLS	CM	AD	RAD	P
25	Mean	$\hat{\theta}$	0.23288	0.20925	0.22713	0.22995	0.22030	0.22600	0.23761	0.26666
		$\hat{\lambda}$	0.42254	0.33824	0.41173	0.42693	0.39962	0.41738	0.43526	0.45268
	MSE	$\hat{\theta}$	0.00493^{6}	0.00382^{2}	0.00418^{4}	0.00436^{5}	0.00373^{1}	0.00392^{3}	0.00512^{7}	0.00864^{8}
		$\hat{\lambda}$	0.07195^{8}	0.04906^{3}	0.04743^{2}	0.05323^{4}	0.04265^{1}	0.05923^{5}	0.06918^{7}	0.06518^{6}
	RBIAS	$\hat{\theta}$	0.16439	0.04624	0.13567	0.14973	0.10149	0.13001	0.18804	0.33332
		$\hat{\lambda}$	0.69018	0.35296	0.64691	0.70774	0.59849	0.66925	0.74104	0.81072
	Σ *Ranks*		14^{7}	5^{2}	6^{3}	9^{5}	2^{1}	8^{4}	14^{7}	14^{7}
50	Mean	$\hat{\theta}$	0.22645	0.20807	0.21918	0.22139	0.21558	0.22125	0.22866	0.25584
		$\hat{\lambda}$	0.41035	0.33104	0.39376	0.40707	0.38834	0.41349	0.42199	0.48058
	MSE	$\hat{\theta}$	0.00293^{7}	0.00215^{4}	0.00208^{3}	0.00216^{5}	0.00167^{1}	0.00200^{2}	0.00265^{6}	0.00513^{8}
		$\hat{\lambda}$	0.06923^{7}	0.04718^{3}	0.04594^{1}	0.05390^{4}	0.04638^{2}	0.05687^{5}	0.06319^{6}	0.08291^{8}
	RBIAS	$\hat{\theta}$	0.13223	0.04033	0.09589	0.10694	0.07788	0.10627	0.14332	0.27922
		$\hat{\lambda}$	0.64138	0.32417	0.57504	0.62829	0.55337	0.65396	0.68797	0.92231
	Σ *Ranks*		14^{7}	$7^{3.5}$	4^{2}	9^{5}	3^{1}	$7^{3.5}$	12^{6}	16^{8}
75	Mean	$\hat{\theta}$	0.22131	0.20642	0.21543	0.21627	0.21234	0.21538	0.22160	0.24571
		$\hat{\lambda}$	0.39697	0.32288	0.39151	0.39221	0.37939	0.38951	0.40050	0.45863
	MSE	$\hat{\theta}$	0.00216^{7}	0.00165^{5}	0.00130^{2}	0.00136^{3}	0.00118^{1}	0.00139^{4}	0.00182^{6}	0.00377^{8}
		$\hat{\lambda}$	0.06217^{7}	0.04355^{2}	0.04390^{3}	0.04907^{4}	0.04063^{1}	0.04964^{5}	0.05608^{6}	0.07641^{8}
	RBIAS	$\hat{\theta}$	0.10654	0.03208	0.07717	0.08134	0.06172	0.07692	0.10798	0.22855
		$\hat{\lambda}$	0.58787	0.29152	0.56606	0.56883	0.51756	0.55804	0.60200	0.83451
	Σ *Ranks*		14^{7}	$7^{3.5}$	5^{2}	$7^{3.5}$	2^{1}	9^{5}	12^{6}	16^{8}
100	Mean	$\hat{\theta}$	0.22434	0.21180	0.21881	0.21987	0.21690	0.21828	0.22446	0.24596
		$\hat{\lambda}$	0.39265	0.32959	0.37828	0.38357	0.37207	0.37828	0.39072	0.45797
	MSE	$\hat{\theta}$	0.00213^{7}	0.00144^{4}	0.00140^{2}	0.00155^{5}	0.00136^{1}	0.00142^{3}	0.00186^{6}	0.00356^{8}
		$\hat{\lambda}$	0.06494^{7}	0.04636^{3}	0.04072^{1}	0.04797^{5}	0.04254^{2}	0.04682^{4}	0.05478^{6}	0.08066^{8}

Table 1. *Cont.*

n	Est.	Est. ; Par.	MLE	MPS	LS	WLS	CM	AD	RAD	P
	RBIAS	$\hat{\theta}$	0.12169	0.05900	0.09406	0.09935	0.08449	0.09139	0.12230	0.22981
		$\hat{\lambda}$	0.57061	0.31835	0.51312	0.53427	0.48829	0.51313	0.56288	0.83187
	Σ *Ranks*		14^{7}	$7^{3.5}$	$3^{1.5}$	10^{5}	$3^{1.5}$	$7^{3.5}$	12^{6}	16^{8}

Table 2. Mean, MSE, and Rbias for several estimation techniques of TTE model at different sample size and θ = 0.2 and λ = 0.5.

n	Est.	Est. ; Par.	MLE	MPS	LS	WLS	CM	AD	RAD	P
25	Mean	$\hat{\theta}$	0.21222	0.21382	0.22744	0.22993	0.23519	0.23034	0.22972	0.23665
		$\hat{\lambda}$	0.54591	0.62554	0.62387	0.63505	0.67390	0.64667	0.64529	0.52600
	MSE	$\hat{\theta}$	0.00416^{1}	0.00468^{2}	0.00606^{7}	0.00602^{5}	0.00702^{8}	0.00603^{6}	0.00487^{4}	0.00473^{3}
		$\hat{\lambda}$	0.09267^{4}	0.14387^{8}	0.08561^{2}	0.08781^{3}	0.11489^{7}	0.09713^{6}	0.09279^{5}	0.04487^{1}
	RBIAS	$\hat{\theta}$	0.06109	0.06912	0.13721	0.14965	0.17596	0.15171	0.14861	0.18327
		$\hat{\lambda}$	0.09181	0.25108	0.24774	0.27010	0.35860	0.29335	0.29058	0.05201
	Σ *Ranks*		5^{2}	10^{6}	$9^{4.5}$	8^{3}	15^{8}	12^{7}	$9^{4.5}$	4^{1}
50	Mean	$\hat{\theta}$	0.21160	0.20777	0.22620	0.22817	0.23148	0.22951	0.22475	0.22670
		$\hat{\lambda}$	0.54773	0.55913	0.64139	0.64894	0.67529	0.65663	0.62742	0.52824
	MSE	$\hat{\theta}$	0.00274^{2}	0.00334^{3}	0.00443^{6}	0.00442^{5}	0.00516^{8}	0.00452^{7}	0.00343^{4}	0.00248^{1}
		$\hat{\lambda}$	0.07786^{2}	0.11370^{8}	0.08786^{3}	0.08818^{4}	0.10958^{7}	0.09425^{6}	0.08884^{5}	0.03364^{1}
	RBIAS	$\hat{\theta}$	0.05801	0.03886	0.0.13102	0.14087	0.15738	0.14756	0.12376	0.13350
		$\hat{\lambda}$	0.09546	0.11827	0.28278	0.29787	0.35058	0.31326	0.25483	0.05647
	Σ *Ranks*		4^{2}	11^{6}	9^{4}	9^{4}	10^{8}	13^{7}	9^{4}	2^{1}
75	Mean	$\hat{\theta}$	0.20947	0.20258	0.22376	0.22342	0.22886	0.22381	0.21985	0.21992
		$\hat{\lambda}$	0.52016	0.50522	0.61961	0.61511	0.65176	0.61953	0.59474	0.51032
	MSE	$\hat{\theta}$	0.00234^{2}	0.00256^{3}	0.00364^{7}	0.00348^{5}	0.00424^{8}	0.00349^{6}	0.00259^{4}	0.00180^{1}
		$\hat{\lambda}$	0.07677^{2}	0.09727^{7}	0.08128^{4}	0.08195^{5}	0.09878^{8}	0.08443^{6}	0.07935^{3}	0.03897^{1}
	RBIAS	$\hat{\theta}$	0.04736	0.01288	0.11881	0.11712	0.14429	0.11905	0.09923	0.09958
		$\hat{\lambda}$	0.04032	0.01045	0.23922	0.23022	0.30353	0.23905	0.18948	0.02065

Table 2. *Cont.*

n	Est.	Est. ; Par.	MLE	MPS	LS	WLS	CM	AD	RAD	P
	Σ *Ranks*		4^{2}	$10^{4.5}$	11^{6}	$10^{4.5}$	16^{8}	12^{7}	7^{3}	2^{1}
100	Mean	$\hat{\theta}$	0.21403	0.20531	0.22717	0.22754	0.23142	0.22821	0.22378	0.21984
		$\hat{\lambda}$	0.55565	0.51409	0.64486	0.64554	0.67276	0.64894	0.62335	0.53714
	MSE	$\hat{\theta}$	0.00241^{2}	0.00253^{3}	0.00357^{7}	0.00346^{5}	0.00400^{8}	0.00354^{6}	0.00276^{4}	0.00157^{1}
		$\hat{\lambda}$	0.07522^{2}	0.09829^{7}	0.08945^{5}	0.08546^{4}	0.10516^{8}	0.09027^{6}	0.08369^{3}	0.03618^{1}
	RBIAS	$\hat{\theta}$	0.07013	0.02655	0.13585	0.13772	0.15712	0.14104	0.11891	0.09921
		$\hat{\lambda}$	0.11130	0.02818	0.28971	0.29108	0.34551	0.29789	0.24669	0.07429
	Σ *Ranks*		4^{2}	10^{5}	$12^{6.5}$	9^{4}	16^{8}	$12^{6.5}$	7^{3}	2^{1}

Table 3. . Mean, MSE, and Rbias for several estimation techniques of TTE model at different sample size and $\theta = 0.2$ and $\lambda = 0.75$.

n	Est.	Est. ; Par.	MLE	MPS	LS	WLS	CM	AD	RAD	P
25	Mean	$\hat{\theta}$	0.21343	0.21447	0.20183	0.20398	0.20962	0.20795	0.21289	0.22035
		$\hat{\lambda}$	0.80465	0.88210	0.68835	0.70728	0.75188	0.74172	0.77118	0.71950
	MSE	$\hat{\theta}$	0.00353^{3}	0.00330^{2}	0.00459^{7}	0.00425^{5}	0.00492^{8}	0.00427^{6}	0.00386^{4}	0.00294^{1}
		$\hat{\lambda}$	0.06478^{5}	0.06961^{6}	0.07103^{7}	0.06433^{4}	0.07210^{8}	0.06124^{3}	0.05369^{2}	0.04747^{1}
	RBIAS	$\hat{\theta}$	0.06716	0.07237	0.00915	0.01988	0.04812	0.03975	0.06444	0.10177
		$\hat{\lambda}$	0.07286	0.17613	0.08220	0.05695	0.00251	0.01104	0.02824	0.04067
	Σ *Ranks*		$8^{5.5}$	$8^{5.5}$	14^{7}	$9^{3.5}$	16^{8}	$9^{3.5}$	6^{2}	2^{1}
50	Mean	$\hat{\theta}$	0.20943	0.21342	0.19848	0.20162	0.20412	0.20330	0.20721	0.21528
		$\hat{\lambda}$	0.77107	0.83535	0.68422	0.70616	0.72938	0.72069	0.74601	0.72274
	MSE	$\hat{\theta}$	0.00249^{2}	0.00251^{3}	0.00329^{7}	0.00309^{6}	0.00343^{8}	0.00306^{5}	0.00255^{4}	0.00184^{1}
		$\hat{\lambda}$	0.06045^{4}	0.06996^{6}	0.07000^{7}	0.06141^{5}	0.07019^{9}	0.06004^{3}	0.05180^{2}	0.03942^{1}
	RBIAS	$\hat{\theta}$	0.04716	0.06712	0.00759	0.00809	0.02059	0.01649	0.03605	0.07640
		$\hat{\lambda}$	0.02809	0.11380	0.08771	0.05845	0.02749	0.03907	0.00532	0.03635
	Σ *Ranks*		$6^{2.5}$	9^{5}	14^{7}	11^{6}	16^{8}	8^{4}	$6^{2.5}$	2^{1}

Table 3. *Cont.*

n	Est.	Est. ; Par.	MLE	MPS	LS	WLS	CM	AD	RAD	P
75	Mean	$\hat{\theta}$	0.20622	0.21216	0.19873	0.20219	0.20426	0.20351	0.20561	0.21227
		$\hat{\lambda}$	0.75291	0.81449	0.69372	0.72012	0.73655	0.73194	0.74534	0.72862
	MSE	$\hat{\theta}$	0.00186^{2}	0.00218^{6}	0.00243^{7}	0.00217^{5}	0.00252^{8}	0.00214^{4}	0.00188^{3}	0.00132^{1}
		$\hat{\lambda}$	0.05660^{4}	0.06770^{6}	0.07017^{8}	0.05788^{5}	0.06779^{7}	0.05641^{3}	0.04946^{2}	0.03306^{1}
	RBIAS	$\hat{\theta}$	0.03111	0.06080	0.00637	0.01097	0.02128	0.01753	0.02805	0.06134
		$\hat{\lambda}$	0.00388	0.08599	0.07505	0.03984	0.01793	0.02409	0.00622	0.02850
	Σ *Ranks*		6^{3}	12^{6}	$15^{7.5}$	10^{5}	$15^{7.5}$	7^{4}	5^{2}	2^{1}
100	Mean	$\hat{\theta}$	0.20117	0.20828	0.19492	0.19892	0.19868	0.19876	0.19990	0.20653
		$\hat{\lambda}$	0.73051	0.79515	0.68971	0.71670	0.71873	0.71578	0.72259	0.71285
	MSE	$\hat{\theta}$	0.00157^{3}	0.00178^{5}	0.00198^{7}	0.00177^{4}	0.00205^{8}	0.00179^{6}	0.00156^{2}	0.00103^{1}
		$\hat{\lambda}$	0.05760^{5}	0.06706^{7}	0.06698^{6}	0.05468^{3}	0.06814^{8}	0.05674^{4}	0.04965^{2}	0.03092^{1}
	RBIAS	$\hat{\theta}$	0.00587	0.04138	0.02539	0.00542	0.00659	0.00618	0.00052	0.03266
		$\hat{\lambda}$	0.02599	0.06020	0.08039	0.04440	0.04169	0.04563	0.03655	0.04954
	Σ *Ranks*		8^{4}	12^{6}	13^{7}	7^{3}	16^{8}	10^{5}	4^{2}	2^{1}

$$K = \frac{E(X^4) - 4E(X)E(X^3) + 6E(X^2)\left(E(X)\right)^2 - 3(E(X))^4}{(V(X))^2}.$$

3.2. Moment Generating Function

The moment generating function and moments provide valuable insights into the properties and features of a distribution. Suppose that X is a random variable with the PDF (3) then the moment generating function of X, $M_X(t)$ Equation (6).

$$\begin{aligned} M_X(t) = E(e^{tX}) &= \int_0^1 e^{tx}\, f_{TTE}(x;\theta,\lambda)\, dx \\ &= \frac{A}{\theta}\int_0^1 e^{\left(t-\frac{1}{\theta}\right)x}\left[1-\lambda+2\lambda e^{-\frac{x}{\theta}}\right] dx \\ &= \frac{A}{\theta}\left[\left(\frac{1-\lambda}{t-\frac{1}{\theta}}\right)\left(e^{t-\frac{1}{\theta}}-1\right)+\frac{2\lambda}{t-\frac{2}{\theta}}\left(e^{t-\frac{2}{\theta}}-1\right)\right]. \end{aligned} \quad \textbf{(6)}$$

3.3. Incomplete Moments and Inequality Measures

In several statistical disciplines, incomplete moments are frequently used, especially in income model analysis to measure inequality. The r^{th} incomplete moment of the TTE distribution can be obtained as Equation (7).

$$\begin{aligned} \varphi_s(t) &= \int_0^t x^s\, f_{TTE}(x;\theta,\lambda)\, dx = \frac{A}{\theta}\int_0^t x^s\, e^{-\frac{x}{\theta}}\left[1-\lambda+2\lambda e^{-\frac{x}{\theta}}\right] dx, \\ &= A\theta^S\left[(1-\lambda)\left[\Gamma(s+1)-\Gamma\left(s+1,\frac{t}{\theta}\right)\right]+\lambda 2^{-s}\left[\Gamma(s+1)\right.\right. \\ &\quad \left.\left.-\Gamma\left(s+1,\frac{2t}{\theta}\right)\right]\right]. \end{aligned}$$

Lorenz $[L_F(z)]$ and Bonferroni $[B_F(z)]$ curves are commonly used income inequality measurements that can be used to various different fields. Mathematically, the $[L_F(z)]$ and $[B_F(z)]$ formulas can be determined as:

$$\begin{aligned} L_F(z) &= \frac{\int_0^z x f_{TTE}(x;\theta,\lambda)dx}{E(X)} \\ &= \frac{\left[(1-\lambda)\left[1-\Gamma\left(2,\frac{z}{\theta}\right)\right]+\frac{\lambda}{2}\left[1-\Gamma\left(2,\frac{2z}{\theta}\right)\right]\right]}{\left[(1-\lambda)\left[1-\Gamma\left(2,\frac{1}{\theta}\right)\right]+\frac{\lambda}{2}\left[1-\Gamma\left(2,\frac{2}{\theta}\right)\right]\right]}, \end{aligned}$$

and

$$\begin{aligned} B_F(z) &= \frac{L_F(z)}{F_{TTE}(z;\theta,\lambda)} \\ &= \frac{\left[(1-\lambda)\left[1-\Gamma\left(2,\frac{z}{\theta}\right)\right]+\frac{\lambda}{2}\left[1-\Gamma\left(2,\frac{2z}{\theta}\right)\right]\right]}{A\left[(1-\lambda)\left[1-\Gamma\left(2,\frac{1}{\theta}\right)\right]+\frac{\lambda}{2}\left[1-\Gamma\left(2,\frac{2}{\theta}\right)\right]\right]\left[\left[1-e^{-\frac{z}{\theta}}\right]\left[1+\lambda e^{-\frac{z}{\theta}}\right]\right]}. \end{aligned}$$

3.4. Quantile Function

Let X denotes a random variable with the PDF (3). The quantile function x_p of the TTE distribution can be obtained as the inverse of CDF (4) by using the Equation (8).

$$A\left[1-e^{-\frac{x}{\theta}}\right]\left[1+\lambda e^{-\frac{x}{\theta}}\right] = p. \quad \textbf{(8)}$$

Then, we have

$$x_p = -\theta \ln\left[\frac{\lambda - 1 + \sqrt{(\lambda+1)^2 - \frac{4\lambda p}{A}}}{2\lambda}\right]. \quad \textbf{(9)}$$

The median of the TTE distribution can be obtained from Equation (9) at $p = 0.5$ as,

$$x_{0.5} = -\theta \ln\left[\frac{\lambda - 1 + \sqrt{(\lambda+1)^2 - \frac{2\lambda}{A}}}{2\lambda}\right].$$

3.5. Mean Residual Life Function

The mean residual life (MRL) refers to the predicted duration of existence, denoted as $X - x$, under the condition that the entity has endured till time x.. The MRL function for the truncated transmuted exponential distribution is defined by Equation (10).

$$\begin{aligned} m(x) = E(X-x|X>x) &= \frac{1}{1-F(x)}\int_x^1 [1-F(t)]dt, \quad \textbf{(10)} \\ &= \frac{1}{1-\frac{\left[1-e^{-\frac{x}{\theta}}\right]\left[1+\lambda e^{-\frac{x}{\theta}}\right]}{\left[1-e^{-\frac{1}{\theta}}\right]\left[1+\lambda e^{-\frac{1}{\theta}}\right]}}\int_x^1 1-\frac{\left[1-e^{-\frac{t}{\theta}}\right]\left[1+\lambda e^{-\frac{t}{\theta}}\right]}{\left[1-e^{-\frac{1}{\theta}}\right]\left[1+\lambda e^{-\frac{1}{\theta}}\right]}, \\ &= \frac{(1-x)\left[1-e^{-\frac{1}{\theta}}\right]\left[1+\lambda e^{-\frac{1}{\theta}}\right]-(1-x)+\theta(\lambda-1)\left[e^{-\frac{1}{\theta}}-e^{-\frac{x}{\theta}}\right]-\frac{\lambda\theta}{2}\left[e^{-\frac{2}{\theta}}-e^{-\frac{2x}{\theta}}\right]}{\left[1-e^{-\frac{1}{\theta}}\right]\left[1+\lambda e^{-\frac{1}{\theta}}\right]-\left[1-e^{-\frac{x}{\theta}}\right]\left[1+\lambda e^{-\frac{x}{\theta}}\right]}. \end{aligned}$$

3.6. Mean Past Lifetime

In practical scenarios where systems are not under constant surveillance, one may want to deduce additional information about the system's past, such as the timing of component failures. Now assume that a component with lifetime X has failed at or some time before x, $x \geq 0$. Consider the conditional random variable $(x - X \mid X \leq x)$. This conditional random variable represents the time has passed since the component's failure, assuming its

Table 4. Mean, MSE, and Rbias for several estimation techniques of TTE model at different sample size and θ = 0.5 and λ = 0.25.

n	Est.	Est. ; Par.	MLE	MPS	LS	WLS	CM	AD	RAD	P
25	Mean	$\hat{\theta}$	0.60034	0.58762	0.59487	0.60017	0.58091	0.58377	0.64173	0.67268
		$\hat{\lambda}$	0.39281	0.41795	0.38695	0.39452	0.42372	0.39707	0.38840	0.36093
	MSE	$\hat{\theta}$	0.22391^{6}	0.27505^{7}	0.21724^{5}	0.19947^{4}	0.15071^{1}	0.19607^{3}	0.19274^{2}	0.30547^{8}
		$\hat{\lambda}$	0.06190^{5}	0.10792^{8}	0.05510^{3}	0.05742^{4}	0.08530^{7}	0.06478^{6}	0.05445^{2}	0.02930^{1}
	RBIAS	$\hat{\theta}$	0.20068	0.17524	0.18974	0.20034	0.16182	0.16754	0.28346	0.34536
		$\hat{\lambda}$	0.57125	0.67182	0.54779	0.57809	0.69488	0.58827	0.55360	0.44373
	Σ *Ranks*		11^{7}	15^{8}	8^{3}	8^{3}	8^{3}	9$^{5.5}$	4^{1}	9$^{5.5}$
50	Mean	$\hat{\theta}$	0.56356	0.54594	0.56286	0.56045	0.55219	0.55410	0.57715	0.60159
		$\hat{\lambda}$	0.35967	0.35065	0.36296	0.35626	0.36413	0.34528	0.33951	0.34674
	MSE	$\hat{\theta}$	0.02674^{3}	0.02477^{1}	0.03064^{6}	0.02962^{5}	0.02658^{2}	0.02897^{4}	0.03437^{7}	0.03758^{8}
		$\hat{\lambda}$	0.04526^{7}	0.04661^{8}	0.04015^{5}	0.03176^{4}	0.04224^{6}	0.02927^{2}	0.03018^{3}	0.02315^{1}
	RBIAS	$\hat{\theta}$	0.12712	0.09187	0.12573	0.12090	0.10439	0.10820	0.15430	0.20318
		$\hat{\lambda}$	0.43868	0.40261	0.45184	0.42504	0.45650	0.38112	0.35804	0.38697
	Σ *Ranks*		10$^{6.5}$	9^{4}	11^{8}	9^{4}	8^{2}	6^{1}	10$^{6.5}$	9^{4}
75	Mean	$\hat{\theta}$	0.56190	0.54938	0.55256	0.55559	0.55023	0.55066	0.57014	0.57587
		$\hat{\lambda}$	0.37430	0.36341	0.37619	0.37809	0.37843	0.36760	0.35907	0.36155
	MSE	$\hat{\theta}$	0.02067^{5}	0.02122^{7}	0.01807^{3}	0.01707^{2}	0.01809^{4}	0.01697^{1}	0.02407^{8}	0.02070^{6}
		$\hat{\lambda}$	0.05180^{6}	0.05501^{7}	0.05080^{5}	0.04027^{4}	0.05556^{8}	0.03872^{3}	0.03130^{2}	0.02957^{1}
	RBIAS	$\hat{\theta}$	0.12379	0.09876	0.10511	0.11118	0.10046	0.10133	0.14028	0.15174
		$\hat{\lambda}$	0.49720	0.45362	0.50478	0.51236	0.51374	0.47042	0.43627	0.44620
	Σ *Ranks*		11^{6}	14^{8}	8^{4}	6^{2}	12^{7}	4^{1}	10^{5}	7^{3}
100	Mean	$\hat{\theta}$	0.55011	0.54265	0.53689	0.53459	0.53220	0.53701	0.54819	0.55785
		$\hat{\lambda}$	0.37267	0.36640	0.36182	0.35231	0.35756	0.35619	0.34998	0.34819
	MSE	$\hat{\theta}$	0.01360^{7}	0.01463^{8}	0.01099^{1}	0.01106^{2}	0.01143^{3}	0.01190^{4}	0.01351^{6}	0.01285^{5}
		$\hat{\lambda}$	0.04915^{7}	0.05657^{8}	0.04134^{5}	0.03390^{3}	0.04167^{6}	0.03528^{4}	0.02880^{2}	0.02332^{1}

Table 4. *Cont.*

n	Est.	Est. ; Par.	MLE	MPS	LS	WLS	CM	AD	RAD	P
	RBIAS	$\hat{\theta}$ $\hat{\lambda}$	0.10023 0.49070	0.08531 0.46559	0.07378 0.44728	0.06918 0.40925	0.06440 0.43025	0.07402 0.42476	0.09638 0.39994	0.11569 0.39276
	Σ *Ranks*		14^{7}	16^{8}	$6^{2.5}$	5^{1}	9^{6}	$8^{4.5}$	$8^{4.5}$	$6^{2.5}$

Table 5. Mean, MSE, and Rbias for several estimation techniques of TTE model at different sample size and θ=0.5 and λ=0.5.

n	Est.	Est. ; Par.	MLE	MPS	LS	WLS	CM	AD	RAD	P
25	Mean	$\hat{\theta}$ $\hat{\lambda}$	0.57802 0.69355	0.57954 0.77922	0.54678 0.60863	0.55778 0.63639	0.56563 0.68572	0.57174 0.65806	0.57359 0.67334	0.57061 0.59708
	MSE	$\hat{\theta}$ $\hat{\lambda}$	0.14483^{8} 0.13322^{6}	0.13943^{7} 0.17365^{8}	0.04100^{3} 0.10623^{2}	0.04193^{4} 0.11297^{3}	0.04353^{5} 0.13474^{7}	0.13863^{6} 0.12397^{5}	0.03739^{2} 0.12232^{4}	0.03393^{1} 0.09588^{1}
	RBIAS	$\hat{\theta}$ $\hat{\lambda}$	0.15604 0.38711	0.15908 0.55844	0.09356 0.21725	0.11556 0.27277	0.13126 0.37145	0.14348 31613	0.14718 0.34667	0.14121 0.19417
	Σ *Ranks*		14^{7}	15^{8}	5^{2}	7^{4}	12^{6}	11^{5}	6^{3}	2^{1}
50	Mean	$\hat{\theta}$ $\hat{\lambda}$	0.54510 0.63160	0.56216 0.71903	0.53782 0.60296	0.54368 0.61352	0.55087 0.64992	0.54737 0.62839	0.56108 0.64908	0.55720 0.59989
	MSE	$\hat{\theta}$ $\hat{\lambda}$	0.02468^{2} 0.11680^{6}	0.02710^{4} 0.15356^{8}	0.02771^{5} 0.10163^{2}	0.02820^{7} 0.10531^{3}	0.02946^{8} 0.12201^{7}	0.02803^{6} 0.11082^{4}	0.02657^{3} 0.11355^{5}	0.02265^{1} 0.09352^{1}
	RBIAS	$\hat{\theta}$ $\hat{\lambda}$	0.09020 0.26319	0.12431 0.43807	0.07564 0.20593	0.08736 0.22704	0.10174 0.29983	0.09474 0.25679	0.12216 0.29817	0.11440 0.19977
	Σ *Ranks*		$8^{3.5}$	12^{7}	7^{2}	$10^{5.5}$	15^{8}	$10^{5.5}$	$8^{3.5}$	2^{1}
75	Mean	$\hat{\theta}$ $\hat{\lambda}$	0.55398 0.65305	0.56833 0.71417	0.54793 0.62507	0.55373 0.64153	0.55981 0.66475	0.55777 0.65374	0.55993 0.65170	0.55335 0.61208
	MSE	$\hat{\theta}$ $\hat{\lambda}$	0.02237^{2} 0.12046^{6}	0.02619^{5} 0.15231^{8}	0.02685^{7} 0.11085^{2}	0.02587^{4} 0.11259^{3}	0.02878^{8} 0.12601^{7}	0.02680^{6} 0.11731^{5}	0.02316^{3} 0.11272^{4}	0.01902^{1} 0.09671^{1}
	RBIAS	$\hat{\theta}$ $\hat{\lambda}$	0.10796 0.30610	0.13666 0.42833	0.09585 0.25014	0.10747 0.28306	0.11962 0.32951	0.11553 0.30748	0.11986 0.30340	0.10670 0.22416

Table 5. *Cont.*

n	Est.	Est. ; Par.	MLE	MPS	LS	WLS	CM	AD	RAD	P
	Σ *Ranks*		8^{4}	13^{7}	9^{5}	$7^{2.5}$	15^{8}	11^{6}	$7^{2.5}$	2^{1}
100	Mean	$\hat{\theta}$	0.55050	0.56512	0.55219	0.55609	0.56277	0.55739	0.55887	0.55229
		$\hat{\lambda}$	0.63954	0.69766	0.62773	0.64299	0.66376	0.64746	0.64836	0.61410
	MSE	$\hat{\theta}$	0.02085^{2}	0.02342^{4}	0.02448^{7}	0.02361^{5}	0.02592^{8}	0.02432^{6}	0.02139^{3}	0.01836^{1}
		$\hat{\lambda}$	0.11928^{6}	0.14394^{8}	0.10949^{2}	0.11146^{3}	0.12248^{7}	0.11403^{5}	0.11400^{4}	0.10103^{1}
	RBIAS	$\hat{\theta}$	0.10100	0.13025	0.10438	0.11218	0.12555	0.11479	0.11773	0.10457
		$\hat{\lambda}$	0.27908	0.39532	0.25546	0.28597	0.32571	0.29493	0.29673	0.22820
	Σ *Ranks*		$8^{3.5}$	12^{7}	9^{5}	$8^{3.5}$	15^{8}	11^{6}	7^{2}	2^{1}

Table 6. . Mean, MSE, and Rbias for several estimation techniques of TTE model at different sample size and θ = 0.5 and λ = 0.75.

n	Est.	Est. ; Par.	MLE	MPS	LS	WLS	CM	AD	RAD	P
25	Mean	$\hat{\theta}$	0.50824	0.52320	0.49146	0.49518	0.50778	0.50244	0.50781	0.50360
		$\hat{\lambda}$	0.73212	0.84598	0.64428	0.65335	0.70819	0.68174	0.69377	0.61976
	MSE	$\hat{\theta}$	0.02797^{3}	0.02777^{2}	0.03455^{5}	0.03487^{7}	0.03597^{8}	0.03464^{6}	0.02819^{4}	0.02328^{1}
		$\hat{\lambda}$	0.08393^{2}	0.07766^{1}	0.09796^{8}	0.09471^{7}	0.09276^{5}	0.09048^{4}	0.08687^{3}	0.09401^{6}
	RBIAS	$\hat{\theta}$	0.01649	0.04640	0.01708	0.00964	0.01557	0.00487	0.01562	0.00721
		$\hat{\lambda}$	0.02384	0.12798	0.14096	0.12886	0.05575	0.09101	0.07498	0.17365
	Σ *Ranks*		5^{2}	3^{1}	$13^{6.5}$	14^{8}	$13^{6.5}$	10^{5}	$7^{3.5}$	$7^{3.5}$
50	Mean	$\hat{\theta}$	0.51859	0.53293	0.50150	0.50972	0.51749	0.51315	0.51860	0.51033
		$\hat{\lambda}$	0.75522	0.83498	0.66839	0.69898	0.72333	0.71017	0.73408	0.68155
	MSE	$\hat{\theta}$	0.01951^{3}	0.01949^{2}	0.02506^{7}	0.02390^{5}	0.02594^{8}	0.02447^{6}	0.02007^{4}	0.01744^{1}
		$\hat{\lambda}$	0.08459^{4}	0.07791^{1}	0.10194^{8}	0.09144^{5}	0.09309^{7}	0.09261^{6}	0.08384^{3}	0.08257^{2}
	RBIAS	$\hat{\theta}$	0.03719	0.06585	0.00299	0.01945	0.03498	0.02630	0.03721	0.02067
		$\hat{\lambda}$	0.00696	0.11330	0.10881	0.06803	0.03556	0.05310	0.02122	0.09127
	Σ *Ranks*		$7^{3.5}$	$3^{1.5}$	$15^{7.5}$	10^{5}	$15^{7.5}$	12^{6}	$7^{3.5}$	$3^{1.5}$

Table 6. *Cont.*

n	Est.	Est. ; Par.	MLE	MPS	LS	WLS	CM	AD	RAD	P
75	Mean	$\hat{\theta}$	0.50097	0.51741	0.48992	0.49663	0.50328	0.49895	0.49871	0.49190
		$\hat{\lambda}$	0.72841	0.80628	0.67248	0.69604	0.71804	0.70606	0.70754	0.66842
	MSE	$\hat{\theta}$	0.01512^{3}	0.01421^{2}	0.01876^{7}	0.01779^{5}	0.01953^{8}	0.01788^{6}	0.01561^{4}	0.01287^{1}
		$\hat{\lambda}$	0.08482^{3}	0.07874^{1}	0.09570^{8}	0.08888^{6}	0.09061^{7}	0.08880^{5}	0.08627^{4}	0.08324^{2}
	RBIAS	$\hat{\theta}$	0.00194	0.03483	0.02017	0.00675	0.00656	0.00210	0.00258	0.01620
		$\hat{\lambda}$	0.02879	0.07504	0.10335	0.07195	0.04261	0.05858	0.05662	0.10877
	Σ *Ranks*		6^{3}	$3^{1.5}$	$15^{7.5}$	$11^{5.5}$	$15^{7.5}$	$11^{5.5}$	8^{4}	$3^{1.5}$
100	Mean	$\hat{\theta}$	0.49355	0.50606	0.48195	0.49060	0.49118	0.49074	0.49268	0.48620
		$\hat{\lambda}$	0.72961	0.78738	0.67746	0.70662	0.71087	0.70818	0.71564	0.67964
	MSE	$\hat{\theta}$	0.01282^{3}	0.01258^{2}	0.01582^{8}	0.01508^{5}	0.01578^{7}	0.01524^{6}	0.01297^{4}	0.01101^{1}
		$\hat{\lambda}$	0.08521^{4}	0.08147^{2}	0.09302^{8}	0.08783^{5}	0.09102^{7}	0.08813^{6}	0.08228^{3}	0.08025^{1}
	RBIAS	$\hat{\theta}$	0.01289	0.01211	0.03609	0.01879	0.01763	0.01851	0.01464	0.02760
		$\hat{\lambda}$	0.02719	0.04985	0.09672	0.05784	0.05217	0.05577	0.04581	0.09381
	Σ *Ranks*		$7^{3.5}$	4^{2}	16^{8}	10^{5}	14^{7}	12^{6}	$7^{3.5}$	2^{1}

Table 7. Partial and total ranks of all estimation techniques for the TTE distribution.

Parameter	n	MLE	MPS	LS	WLS	CM	AD	RAD	P
θ = 0.2, λ = 0.25	25	7	2	3	5	1	4	7	7
	50	7	3.5	2	5	1	3.5	6	8
	75	7	3.5	2	3.5	1	5	6	8
	100	7	3.5	1.5	5	1.5	3.5	6	8
θ = 0.2, λ = 0.5	25	2	6	4.5	3	8	7	4.5	1
	50	2	6	4	4	8	7	4	1
	75	2	4.5	6	4.5	8	7	3	1
	100	2	5	6.5	4	8	6.5	3	1
θ = 0.2, λ = 0.75	25	5.5	5.5	7	3.5	8	3.5	2	1
	50	2.5	5	7	6	8	4	2.5	1
	75	3	6	7.5	5	7.5	4	2	1
	100	4	6	7	3	8	5	2	1
θ = 0.5, λ = 0.25	25	7	8	3	3	3	5.5	1	5.5
	50	6.5	4	8	4	2	1	6.5	4
	75	6	8	4	2	7	1	5	3
	100	7	8	2.5	1	6	4.5	4.5	2.5
θ = 0.5, λ = 0.5	25	7	8	2	4	6	5	3	1
	50	3.5	7	2	5.5	8	5.5	3.5	1
	75	4	7	5	2.5	8	6	2.5	1
	100	3.5	7	5	3.5	8	6	2	1
θ = 0.5, λ = 0.75	25	2	1	6.5	8	6.5	5	3.5	3.5
	50	3.5	1.5	7.5	5	7.5	6	3.5	1.5
	75	3	1.5	7.5	5.5	7.5	5.5	4	1.5
	100	3.5	2	8	5	7	6	3.5	1
∑ Ranks		107.5	119.5	119	100.5	144.5	117	90.5	65.5
Overall Rank		4	7	6	3	8	5	2	1

lifetime is equal to or less than x. Thus, the mean past lifetime (MPL) of the component can be determined as Equation (11).

$$k(x) = E(x - X|X \leq x) = \frac{1}{F(x)} \int_0^x F(t)dt, \quad (11)$$

$$= \frac{1}{\left[1 - e^{-\frac{x}{\theta}}\right]\left[1 + \lambda e^{-\frac{x}{\theta}}\right]} \int_0^x \left[1 + (\lambda - 1)e^{-\frac{t}{\theta}} - \lambda e^{-\frac{2t}{\theta}}\right] dt,$$

$$= \frac{\left[x - \theta(\lambda - 1)\left(e^{-\frac{x}{\theta}} - 1\right) + \frac{\lambda\theta}{2}\left(e^{-\frac{2x}{\theta}} - 1\right)\right]}{\left[1 - e^{-\frac{x}{\theta}}\right]\left[1 + \lambda e^{-\frac{x}{\theta}}\right]}.$$

3.7. Order Statistics

Moments of order statistics are helpful for quality control testing and predicting future failures of items based on a limited number of early failures. We know that if $x_{(1)} \leq \ldots \leq x_{(n)}$ denotes the order statistic of a random sample $x_1, \ldots, x_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$, then the pdf of $X_{(m)}$ is given by Equation (12).

$$f_{x_{(m)}} = \frac{n!}{(m-1)!\,(n-m)!} f_X(x)\left(F_X(x)\right)^{m-1}\left(1 - F_X(x)\right)^{n-m} \quad (12)$$

The pdf of the m^{th} order statistic for the TTE distribution is offered by Equation (13).

$$f_{x_{(m)}} = \frac{n!}{(m-1)!\,(n-m)!} \frac{A}{\theta} e^{-\frac{x}{\theta}}\left[1 - \lambda + 2\lambda e^{-\frac{x}{\theta}}\right]$$

$$\times \left[A\left[1 - e^{-\frac{x}{\theta}}\right]\left[1 + \lambda e^{-\frac{x}{\theta}}\right]\right]^{m-1} \times \left[1 - A\left[1 - e^{-\frac{x}{\theta}}\right]\left[1 + \lambda e^{-\frac{x}{\theta}}\right]\right]^{n-m}. \quad (13)$$

Then, the pdf of the greatest order statistic $X_{(n)}$ is evaluated by Equation (14),

$$f_{x_{(n)}} = \frac{nA}{\theta} e^{-\frac{x}{\theta}} \left[1 - \lambda + 2\lambda e^{-\frac{x}{\theta}}\right] \times \left[A\left[1 - e^{-\frac{x}{\theta}}\right]\left[1 + \lambda e^{-\frac{x}{\theta}}\right]\right]^{n-1}, \quad (14)$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by Equation (15).

$$f_{x_{(1)}} = \frac{nA}{\theta} e^{-\frac{x}{\theta}} \left[1 - \lambda + 2\lambda e^{-\frac{x}{\theta}}\right] \times \left[1 - A\left[1 - e^{-\frac{x}{\theta}}\right]\left[1 + \lambda e^{-\frac{x}{\theta}}\right]\right]^{n-1} \quad (15)$$

3.8. Extropy

Extropy, a different measure of uncertainty, was recently introduced. The definition of extropy for a completely continuous, non-negative random variable X is:

$$J(x) = \frac{-1}{2}\int_0^{\infty} \left(f(x)\right)^2 dx$$

$$= \frac{-1}{2}\int_0^{\infty} \left(\frac{A}{\theta} e^{-\frac{x}{\theta}} \left[1 - \lambda + 2\lambda e^{-\frac{x}{\theta}}\right]\right)^2 dx$$

$$= \frac{-A^2}{2\theta^2}\int_0^{\infty} \left(e^{-\frac{x}{\theta}} \left[1 - \lambda + 2\lambda e^{-\frac{x}{\theta}}\right]\right)^2 dx$$

$$= \frac{-A^2}{12\theta}(\lambda^2 + 2\lambda + 3), \quad \theta > 0.$$

4. METHODS OF ESTIMATION

This section covers eight methods for estimating the parameters, θ and λ of the TTE distribution. The methods include ML, LS, WLS, MPS, CM, AD, RAD and P. We suppose throughout that $x_1, x_2, \ldots, x_n$ represents a random sample of size n from the TTE model.

4.1. Method of Maximum Likelihood Estimation

The ML approach was used to estimate the parameters of the TTE distribution. The likelihood function, L can be obtained as Equation (16).

$$L = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \left(\frac{A}{\theta} e^{-\frac{x_i}{\theta}} \left[1 - \lambda + 2\lambda e^{-\frac{x_i}{\theta}}\right]\right). \quad (16)$$

Therefore, the log of the likelihood function can be given as the form:

$$\ln(L) = n\ln(A) - n\ln(\theta) - \frac{1}{\theta}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln\left(1 - \lambda + 2\lambda e^{-\frac{x_i}{\theta}}\right).$$

Then, the first derivatives of the log likelihood function with respect to the parameters θ and λ are determined as Equation (17):

$$\frac{\partial \ln(L)}{\partial\theta} = \frac{n}{A}\frac{\partial A}{\partial\theta} - \frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n}(x_i) + \frac{2\lambda}{\theta^2}\sum_{i=1}^{n}\frac{x_i e^{-\frac{x_i}{\theta}}}{\left(1 - \lambda + 2\lambda e^{-\frac{x_i}{\theta}}\right)}, \quad (17)$$

$$\frac{\partial \ln(L)}{\partial\lambda} = \frac{n}{A}\frac{\partial A}{\partial\lambda} + \sum_{i=1}^{n}\frac{2e^{-\frac{x_i}{\theta}} - 1}{\left(1 - \lambda + 2\lambda e^{-\frac{x_i}{\theta}}\right)}.$$

where,

$$\frac{\partial A}{\partial\theta} = \frac{\frac{1}{\theta^2}\left(1 + \lambda e^{-\frac{1}{\theta}}\right)e^{-\frac{1}{\theta}} - \frac{\lambda}{\theta^2}\left(1 - e^{-\frac{1}{\theta}}\right)e^{-\frac{1}{\theta}}}{\left(\left[1 - e^{-\frac{1}{\theta}}\right]\left[1 + \lambda e^{-\frac{1}{\theta}}\right]\right)^2},$$

$$\frac{\partial A}{\partial\lambda} = \frac{-\left(1 - e^{-\frac{1}{\theta}}\right)e^{-\frac{1}{\theta}}}{\left(\left[1 - e^{-\frac{1}{\theta}}\right]\left[1 + \lambda e^{-\frac{1}{\theta}}\right]\right)^2}.$$

For obtaining the estimates of the parameters θ and λ, utilize numerical methods and computer resources to solve the equations of $\frac{\partial \ln(L)}{\partial\theta} = 0$ and $\frac{\partial \ln(L)}{\partial\lambda} = 0$.

Table 8. Performance of several distributions forms for the first data set.

Model	MLE	AIC	BIC	CAIC	HQIC	K-S	P-value
TTE	$\hat{\theta} = 0.8778$ $\hat{\lambda} = 0.3348$	-0.7889	1.6487	-0.2435	-0.1128	0.074	0.997
TME	$\hat{\alpha} = 0.2087$	6.118	7.3377	6.2927	6.4569	0.136	0.693
Ex	$\hat{\lambda} = 2.662$	3.048	4.2664	3.2214	3.3855	0.111	0.883
TL	$\hat{\lambda} = 1.002$	6.461	7.6799	6.635	6.7991	0.117	0.844
Gex	$\hat{\alpha} = 1.151$ $\hat{\lambda} = 0.3444$	4.770	7.208	5.316	5.447	0.089	0.979

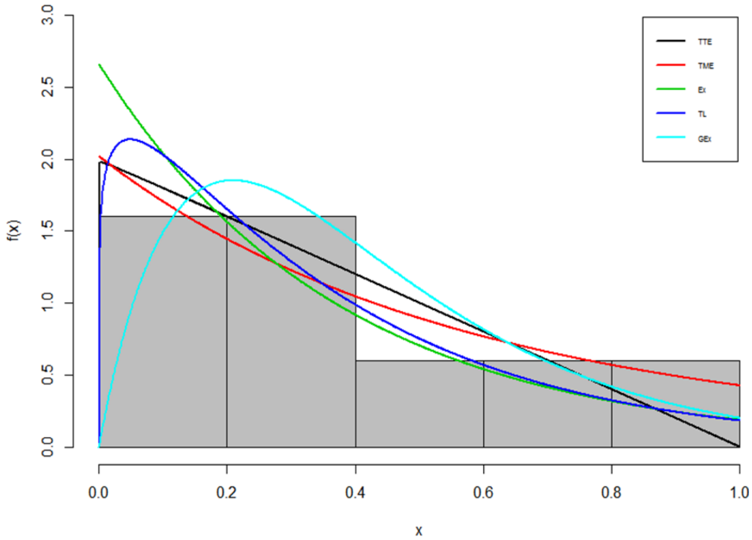

Figure 3. Plot of the estimated PDF of the first data set of the selected models.

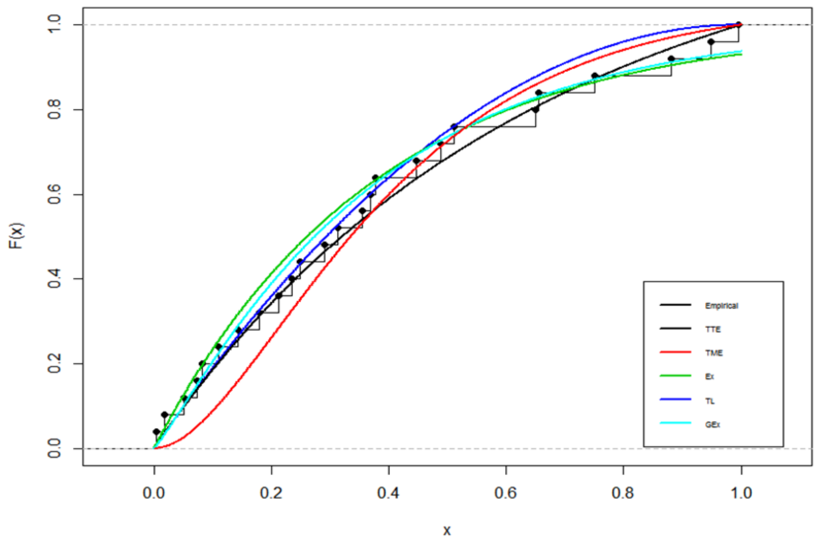

Figure 4. Plot of the estimated CDF of the first data set of the selected models.

4.2. Method of Least Square and Weighted Least Square Estimation

The LS approach is a statistical technique used to determine the optimal fit for a dataset by minimizing the total sum of the differences between data points and the curve being fitted. Swain et al.; Onyekwere et al.; and Husain et al. presented least square estimators and weighted least square estimators for estimating the parameters of Beta distributions [33]-[35]. We utilize the identical technique for the TTE distribution in this investigation. The LS estimators for the parameters θ and λ of the TTE distribution are derived by minimizing:

$$\sum_{j=1}^{n}\left[F(X_{(j)})-\frac{j}{n+1}\right]^2,$$

with respect to the unknown parameters θ and λ. Let $F(X_{(j)})$ represents the distribution function of the arranged random variables $X_{(1)} < X_{(2)} < \dots < X_{(n)}$, where $\{X_1, X_2, \dots X_n\}$ is a random sample of size n from a CDF $F(.)$.Hence, in this case, the LS estimates of θ and λ (say, $\hat{\theta}_{LSE}$ and $\hat{\lambda}_{LSE}$) can be found by minimizing:

$$\sum_{j=1}^{n}\left[A\left[1-e^{-\frac{x_{(i)}}{\theta}}\right]\left[1+\lambda e^{-\frac{x_{(i)}}{\theta}}\right]-\frac{j}{n+1}\right]^2,$$

with respect to θ and λ. The WLS estimates of the unknown parameters θ and λ can be found by minimizing [36][37].

$$\sum_{j=1}^{n}w_j\left[F(X_{(j)})-\frac{j}{n+1}\right]^2,$$

with respect to θ and λ . The weights w_j are equal to $\frac{1}{V(x_{(j)})}=\frac{(n+1)^2(n+2)}{j(n-j+1)}$.

Hence, in this case, the WLS estimations of θ and λ (denoted as $\hat{\theta}_{WLSE}$ and $\hat{\lambda}_{WLSE}$) respectively, can be determined by minimizing:

$$\sum_{j=1}^{n}\frac{(n+1)^2(n+2)}{n-j+1}\left[A\left[1-e^{-\frac{x_{(i)}}{\theta}}\right]\left[1+\lambda e^{-\frac{x_{(i)}}{\theta}}\right]-\frac{j}{n+1}\right]^2,$$

with respect to θ and λ.

4.3. Method of Maximum Product of Spacing Estimation

Cheng and Amin developed a sophisticated method, known as the MPS method, to calculate the estimation of unknown parameters in continuous univariate distributions [38][39]. Ranneby separately created it as an approximation to the Kullback-Leibler measure of information [40]. The basic concept can be explained as follows:

Let $D_i(\theta,\lambda)=F(x_{i=1:n}|\theta,\lambda)-F(x_{i-1:n}|\theta,\lambda)$, for $i=1,2,\ldots\ldots,n+1$, represent the uniform spacings of a random sample from the TTE distribution, where $F(x_{0:n}|\theta,\lambda)=0$, $F(x_{n+1:n}|\theta,\lambda)=1$ and $\sum_{i=1}^{n+1}D_i(\theta,\lambda)=1$.

The maximum product of spacing estimator (MPSE) for $\hat{\theta}_{MPSE}$ and $\hat{\lambda}_{MPSE}$ can be determined by maximizing the geometric mean of the spacings

$$G(\theta,\lambda)=\left[\prod_{i=1}^{n+1}D_i(\theta,\lambda)\right]^{\frac{1}{n+1}},$$

Equally, they can be obtained by maximizing the logarithm of the geometric mean of sample spacings:

$$H(\theta,\lambda)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i(\theta,\lambda).$$

The estimates $\hat{\theta}_{MPSE}$ and $\hat{\lambda}_{MPSE}$ can be obtained by solving the two nonlinear equations:

$$\frac{\partial H(\theta,\lambda)}{\partial\theta}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\theta,\lambda)}[\Delta_1(x_{i:n}|\theta,\lambda)-\Delta_1(x_{i-1:n}|\theta,\lambda)]=0,$$

and,

$$\frac{\partial H(\theta,\lambda)}{\partial\lambda}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\theta,\lambda)}[\Delta_2(x_{i:n}|\theta,\lambda)-\Delta_2(x_{i-1:n}|\theta,\lambda)]=0,$$

where,

$$\Delta_1(x_{i:n}|\theta,\lambda)=\frac{\partial F(\theta,\lambda)}{\partial\theta},\qquad \Delta_2(x_{i:n}|\theta,\lambda)=\frac{\partial F(\theta,\lambda)}{\partial\lambda}\quad\textbf{(18)}$$

Cheng and Amin [39] suggest that MPS estimates exhibit consistency across a broader array of scenarios compared to MLE estimators. Additionally, maximizing H for parameter estimation is equally efficient like MLE estimation.

4.4. Method of Cramér-Von-Mises Estimation

CM estimation refers to the discrepancy between the estimated CDF and its empirical distribution function. Based on actual data from MacDonald [41][42], the bias of the estimations is lesser compared to other minimal distance estimates. The

Table 9. Performance of several distribution forms for the second data set.

Model	MLE	AIC	BIC	CAIC	HQIC	K-S	P-value
TTE	$\hat{\theta}=0.1317$ $\hat{\lambda}=0.5634$	-77.416	-74.614	-76.972	-76.519	0.087	0.978
TME	$\hat{\alpha}=0.0481$	-65.792	-64.391	-65.649	-65.344	0.237	0.0691
We	$\hat{\alpha}=0.9394$ $\hat{\lambda}=0.0932$	-76.717	-73.915	-76.273	-75.821	0.101	0.919
Gamma	$\hat{\alpha}=0.9688$ $\hat{\lambda}=0.0993$	-76.520	-73.717	-76.076	-75.624	0.106	0.892
Beta	$\hat{\alpha}=0.8576$ $\hat{\beta}=7.806$	-75.121	-72.319	-74.677	-74.225	0.129	0.702

CM estimates $\hat{\theta}_{CME}$ and $\hat{\lambda}_{CME}$ of θ and λ are, respectively, determined by minimizing

$$C(\theta,\lambda) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{i:n}|\theta,\lambda) - \frac{2i-1}{2n}\right]^2,$$

The next nonlinear equations can also be utilized to calculate these estimates:

$$\sum_{i=1}^{n}\left[F(x_{i:n}|\theta,\lambda) - \frac{2i-1}{2n}\right]\Delta_1(x_{i:n}|\theta,\lambda) = 0,$$

$$\sum_{i=1}^{n}\left[F(x_{i:n}|\theta,\lambda) - \frac{2i-1}{2n}\right]\Delta_2(x_{i:n}|\theta,\lambda) = 0.$$

where, $\Delta_1(.|\theta,\lambda)$ and $\Delta_2(.|\theta,\lambda)$ are given in (Eq.18).

4.5. Method of Anderson-Darling and Right-tail Anderson-Darling Estimation

The AD method, introduced by Anderson and Darling [43], is another sort of minimum distance estimator. The AD estimates (ADE) $\hat{\theta}_{ADE}$ and $\hat{\lambda}_{ADE}$ of the parameters θ and λ are, respectively, obtained by minimizing the following equation.

$$A(\theta,\lambda) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(x_{i:n}|\theta,\lambda) + \log \bar{F}(x_{n+1-i:n}|\theta,\lambda)\right].$$

These estimates can also be determined by solving the following equations:

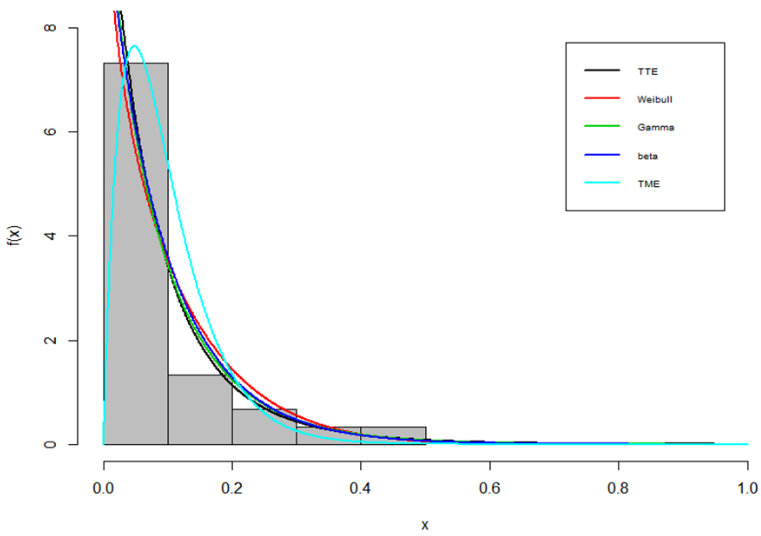

Figure 5. Plot of the estimated PDF of the second data set of the selected models.

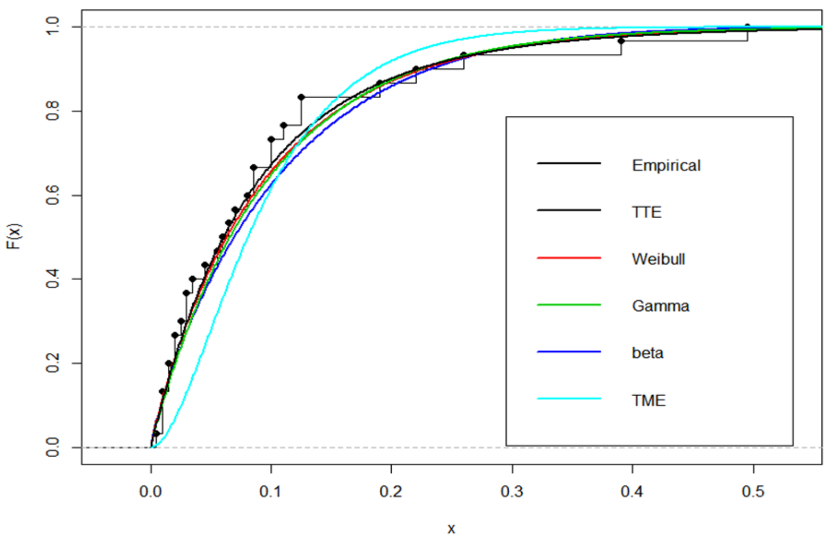

Figure 6. Plot of the estimated CDF of the second data set of the selected models.

Table 10. Performance of several distribution forms for the third data set.

Model	MLE	AIC	BIC	CAIC	HQIC	K-S	P-value
TTE	$\hat{\theta}=0.4628$ $\hat{\lambda}=-0.2529$	-2.6727	0.1296	-2.2283	-1.7762	0.057	0.999
TME	$\hat{\alpha}=0.2008$	-1.1418	0.2594	-0.9989	-0.6935	0.132	0.626
Ex	$\hat{\lambda}=2.7332$	1.6708	3.0721	1.8137	2.1191	0.127	0.670
Gamma	$\hat{\alpha}=1.4852$ $\hat{\lambda}=0.2463$	1.1193	3.9217	1.5638	2.0159	0.103	0.878
Gex	$\hat{\alpha}=1.4906$ $\hat{\lambda}=0.2879$	1.3377	4.1401	1.7822	2.2342	0.106	0.854

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1(x_{i:n}|\theta,\lambda)}{F(x_{i:n}|\theta,\lambda)}-\frac{\Delta_1(x_{n+1-i:n}|\theta,\lambda)}{\bar{F}(x_{n+1-i:n}|\theta,\lambda)}\right]=0\,,$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_2(x_{i:n}|\theta,\lambda)}{F(x_{i:n}|\theta,\lambda)}-\frac{\Delta_2(x_{n+1-i:n}|\theta,\lambda)}{\bar{F}(x_{n+1-i:n}|\theta,\lambda)}\right]=0,$$

where $\Delta_1(.|\theta,\lambda)$ and $\Delta_2(.|\theta,\lambda)$ are given in (Eq.18). Also, the RAD estimates (RADE) $\hat{\theta}_{RTADE}$ and $\hat{\lambda}_{RTADE}$ of the parameters θ and λ are, respectively, obtained by minimizing the following equation.

$$R(\theta,\lambda)=\frac{n}{2}-2\sum_{i=1}^{n}F(x_{i:n}|\theta,\lambda)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\bar{F}(x_{n+1-i:n}|\theta,\lambda).$$

These estimates can be obtained by solving the following Equations (19):

$$-2\sum_{i=1}^{n}\frac{\Delta_1(x_{i:n}|\theta,\lambda)}{F(x_{i:n}|\theta,\lambda)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\Delta_1(x_{n+1-i}|\theta,\lambda)}{\bar{F}(x_{n+1-i}|\theta,\lambda)}=0, \quad \textbf{(19)}$$

where $\Delta_1(.|\theta,\lambda)$ and $\Delta_2(.|\theta,\lambda)$ are given in Equation (18).

4.6. Method of Percentile Estimation

The percentile estimator, initially introduced by Kao [44][45], is a statistical method that estimates unknown parameters by comparing sample points with theoretical ones. The Weibull distribution and the generalized exponential distribution are two distributions with closed-form quantile functions that have been widely used.

Let $u_i=i/((n+1))$ be an unbiased estimator of $F(x_{(i)};\theta,\lambda)$. Then, the PE of the parameters of the TTE distribution is obtained by minimizing the following Equation (20):

$$P(\theta,\lambda)=\sum_{i=1}^{n}\left(x_{(i)}+\theta\ln\left[\frac{\lambda-1+\sqrt{(\lambda+1)^2-\frac{4\lambda u_i}{A}}}{2\lambda}\right]\right)^2. \quad \textbf{(20)}$$

with respect to θ and λ.

5. SIMULATION

The principal purpose of the simulation study is to evaluate the effectiveness of various estimating techniques for the parameters of the TTE model. Estimates are evaluated using relative biases and mean square errors (MSE). The process of simulation is conducted using R software. The procedures of simulation for the TTE model can be assumed as sizes of random samples; *n* = 25,50,75, and 100, are constructed from the TTE model by employing inversion approach. The values of parameters are studied as (θ = 0.2,0.5) and (λ = 0.25,0.5,0.75). The TTE model estimators are assessed according to the values of parameters and sample sizes. We calculate the relative biases and mean squared errors of the estimates for different choices of the model parameters. Tables 1–6 present the empirical outcomes. We note that the values of the relative biases and mean square error decreases with larger sample sizes. The simulation outcomes exhibit the rankings of the estimators for each technique using superscripts in each row, along with the total sum of the ranks denoted by ∑ Ranks. The results in Table 7 show the performance order of all estimators, both individually and collectively. Table 7 indicates that the PE method, with a total score of 65.5, surpasses all estimations from the other techniques for the TTE distribution. The RADE, with a total score of 90.5, could be seen as a rival method to the PE method.

6. APPLICATIONS

We analyze three real data sets to compare the efficacy of the truncated transmuted exponential distribution (TTE) utilizing various criterions for goodness of fit, such as Akaike's information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), and Kolmogorov–Smirnov (K–S) test statistic. The distribution with the lowest AIC, BIC, CAIC, and HQIC values is considered to have the best fit.

6.1. Data set 1

The first data set is used by Kumari et al. [46], originally taken from Proschan [47]. The data represent the intervals between failures (in hours) of the air conditioning system of a fleet of 13 Boeing 720 jet airplanes. Canavos and Tsokos observed that the failure time distribution of the air conditioning system for each of the planes can be well approximated by exponential distributions [48]. We have considered the plane 7913 for our illustrative purposes. The data is presented as follows: 1, 4, 11, 16, 18, 18, 18, 24, 31, 39, 46, 51, 54, 63, 68, 77, 80, 82, 97,106, 111, 141, 142, 163, 191, 206, 216.

Before applying the K–S test, we transform the above given data set in the range of unit interval by

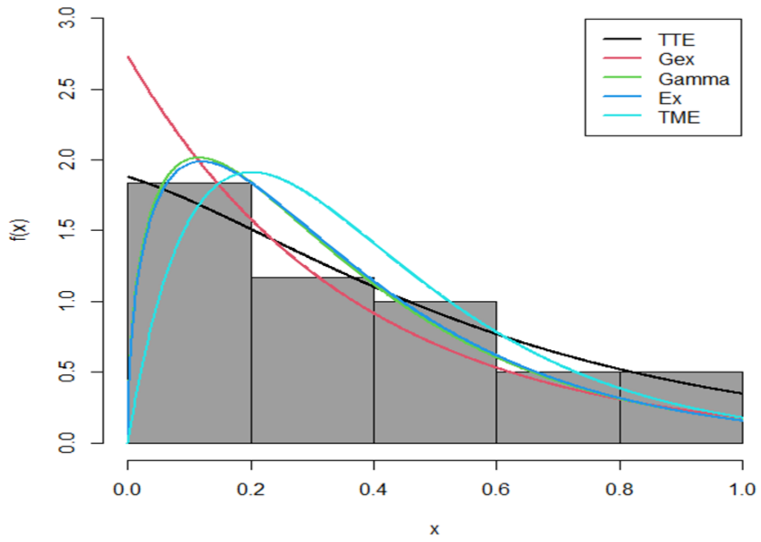

Figure 7. Plot of the estimated PDF of the third data set of the selected models.

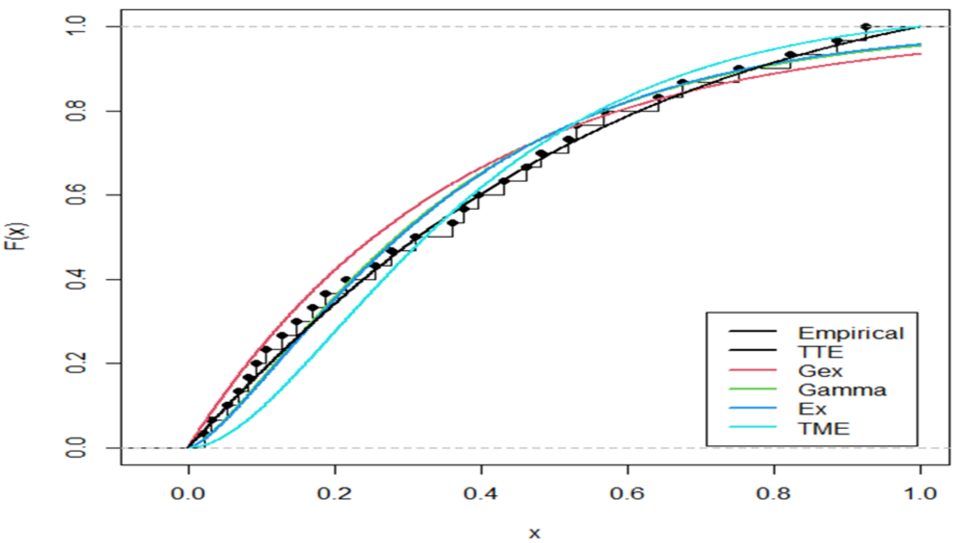

Figure 8. Plot of the estimated CDF of the third data set of the selected models.

using the transformation $x = \frac{x_i}{(max(X_i)+1)}$. The transformed data sets is given as follows: 0.0046, 0.0184, 0.0507, 0.0737, 0.0829, 0.1106, 0.1429, 0.1797, 0.2120, 0.2350, 0.2488, 0.2903, 0.3134, 0.3548, 0.3687, 0.3779, 0.4470, 0.4885, 0.5115, 0.6498, 0.6544, 0.7512, 0.8802, 0.9493, 0.9954.

We can determine the best model for understanding a dataset by evaluating goodness-of-fit measures such as BIC, AIC, HQIC, and K–S test statistic. The lower value of these criteria indicates a better fit. The proposed model is contrasted with exponential (Ex), truncated moment exponential (TME), Topp-Leone (TL) and generalized exponential (Gex), distributions. Table 8 presents the parameter estimates and associated goodness of fit statistics. The data indicate that the TTE model has the lowest K-S value and the highest P-value compared to the other models. Hence, the TTE distribution provides a superior fit to the data compared to all other distributions. Figures 3 and 4 displays the estimated PDF and CDF generated by the new model fitted to this data.

6.2. Data Set 2

The data set relates to the daily snowfall amounts of 30 observations measured in inches of water taken from non-seeded experimental units, which was conducted in the vicinity of Climax, Colorado [49]. The data are presented as follows: 0.030, 0.020, 0.015, 0.045, 0.100, 0.100, 0.125, 0.190, 0.390, 0.110, 0.070, 0.010, 0.055, 0.220, 0.080, 0.005, 0.125, 0.035, 0.085, 0.060, 0.010, 0.065, 0.020, 0.260, 0.030, 0.015, 0.025, 0.010, 0.495, 0.085.

The proposed model is competed with truncated moment exponential (TME), Weibull, Gamma and beta distributions. The outcomes of applying the suggested model to this data set are shown in Table 9. The proposed TTE model shows the smallest values of AIC, BIC, CAIC, and HQIC in comparison to the alternative distributions. Moreover, the P value of the TTE model obtained from the K–S test exceeds those of the chosen distributions. Hence, the TTE distribution gives a better fit to the data compared to all other distributions. Figures 5 and 6 display the estimated PDF and CDF resulting from fitting the suggested model to this data.

6.3. Data Set 3

The third dataset of data included 30 polyester fiber tensile strength measurements that were analyzed using previous data [50][51]. The following is a summary of the data: 0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188, 0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463, 0.481, 0.519, 0.529, 0.567, 0.642, 0.674, 0.752, 0.823, 0.887, 0.926. The truncated moment exponential (TME), exponential (Ex), generalized exponential (Gex), and gamma distributions are the competitors of the suggested model. Table 10 displays the results of using the proposed model on this dataset. Comparing the TTE model to the other distributions, the AIC, BIC, CAIC, and HQIC values are the lowest. Furthermore, the TTE model's P value as determined by the K-S test is higher than the selected distributions. Therefore, when compared to all other distributions, the TTE distribution provides a better fit to the data. The estimated PDF and CDF that come from fitting the proposed model to this data are shown in Figures 7 and 8.

7. CONCLUSION

The article presents a novel statistical model known as the truncated transmuted exponential (TTE) distribution. Several statistical properties including moments, moment generating function, incomplete moments, order statisics, mean residual life function, mean past lifetime function, some measures of reliability and extropy has been studied. The parameters of the newly proposed distribution haven been estimated by eight estimation methods namely, the maximum likelihood, the least squares, weighted least-squares, maximum product of spacing, Crame ́r-Von Mises , Anderson-Darling, right-tail Anderson-Darling and percentile estimators. The results of simulation study showed the superiority of the percentile method among the other estimation methods. The real application showed the importance of the proposed TTE model. Based on some goodness of fit for the suggested data set indicated that the TTE distribution provides consistently better fit than other distributions. Completely, it is well acknowledged that truncated models are effective at simulating real-world problems. So, we suggest utilizing the TTE model in different disciplines like, engineering, health and finance, where truncated data is common.

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Conflicts of Interest

The authors declare no conflict of interest.

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DECLARATION OF GENERATIVE AI

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