A Mathematical Model of an Electoral Process and Predicting of Outcome


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A Mathematical Model of an Electoral Process and Predicting of Outcome

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AUTHOR CONTRIBUTIONS

CONFLICT OF INTEREST

The authors declare no conflict of interest.
A Mathematical Model of an Electoral Process and Predicting of Outcome

Abstract. We developed and analysed a mathematical model to study the dynamics of an electoral process and predict the outcome in Nigeria with three political parties as major actors. We model the ruling party ($P_1$); major opposition party ($P_2$) and minority opposition parties ($P_3$). The model includes party campaigners of the three political parties and a class of eligible voters. The model also incorporates the Independent National Electoral Commission (INEC), electoral observers, judiciary, security personnel and thugs. The influx of voters into the system is determined by the registration rate of voter cards. The model’s interest lies in the measure of positive influence that the party campaigners who eligible voters to join their party. With this, the model mirrors election’s outcome and the movement of voters between the three groups of political parties involved. The model has proven that the pattern of election’s outcome for each political party is affected by the defection of party members and lack of policy consistency. We determine the equilibria analytically and discuss the stability of the system. Numerical simulations are graphed in combination with curve fitting to compare each party’s performances over the years. The result shows that no political party remains in power steadily from 2007 to 2019. Thus, the prediction of elections outcome to the political space of any region in Nigeria depends on political party structures and voters’ decisions.

Keywords: Election; Prediction; Mathematical Model; Eligible Voters; Political Parties; Campaigners; INEC; Judiciary

1. INTRODUCTION

Globally, political elections are more of a chameleon of colours with uncertainties and conflicting interests. A democratic system of government provides a vehicle by which the people and citizens directly retain and exercise political independence [1]. Democracy is thus unique in the fact that the power of government finds its source in the people. The fulcrum point in democracies is competitive elections. Elections are decision-making processes which enable a population to choose an individual to occupy a formal office. In a democracy, offices
in the executive, and parliament legislative arms of government are filled by engaging this mechanism [2].

The study aims to develop and analyse a mathematical model for predicting election’s outcome amongst three groups of political parties in Nigeria. The specific objectives are to obtain the equilibrium states of the model equations; carry out stability analysis of the equilibrium states of the model; obtain the analytical solution of the model equations; determine the bounds of the electoral model; carry out sensitivity analysis on the model parameters and numerical simulations with curve fittings to present the results graphically; and to discuss the research findings in a simple language.

In line with this, we utilised permutations of mathematical applications to study the dynamics of electoral outcomes over the years in Nigeria through modelling and data forecasting. Modelling real life situations using mathematical equations starts with close examination, investigation, and projection. Election’s outcome is one of the real-life problems that is highly sensitive and nonlinear. Therefore, a mathematical model depicting this problem is also nonlinear. The voting system is a single vote method otherwise the voter would be disenfranchised and there are three major groups of political parties. All the political parties contact the voter class to join their party. However, the unwholesome defection of individuals from one political party to the other has become a common trend for many reasons. This is a kind of Monte-Carlo game where some of the party members who do not get proper positions or weight in their party or are not very much satisfied with the changes occurring in the party’s ideology, leave the party and join another one in order to achieve their desire. Thus, election’s outcome is a sensitive process that needs a scientific approach. This research work is essentially a theoretical experiment that provides answers to probing questions on election’s outcome.

Governance and political leadership are a crucial aspect of human existence. In a democratic country like Nigeria, very often, political officeholders and elected parliaments make decisions that are constitutionally binding on all citizens irrespective of ethnicity, religious affiliation or skin colour. In Nigeria, INEC is saddled with the responsibility of conducting elections to make a pronouncement of individuals who win elective positions. The process pivoted eligible voters as important decision-makers. Whether or not it is possible to assess the policy positions of voters and candidates, there are hidden variables that affect a voter’s choice in common cases. Therefore, policy positions are always measured with error, especially in Nigeria democratic setting where political parties lack policy and ideology. A voter’s behaviour depends on either policy component, candidate integrity or imperfect information. Voters normally consider how the policy platform affects their utility such as welfare, cost of living and foreign policy and
also, have sympathy towards a candidate for scandals or feelings. Candidates do not know with certainty the voters sympathy. Voters do not show preferences based on the policy implemented by the winner but also on the candidate’s personality and identity [3]. The average popularity of candidates before the election also affects voter’s decisions. Popularity before elections is out of candidates’ control as the outbreak of scandals or other news may alter the popularity of a candidate, to the advantage of their opponents.

There are many reasons for voters’ uncertainty about the candidates [4]. A candidate policy positions may be imperfectly perceived or may be perceived as a random variable. Uncertainty about new developments and future events may also complicate voter’s decision. Hinich et al [5], they study a theory of electoral equilibrium whereby, future-oriented voters accept unavoidable uncertainty, even if they are confident about the candidate’s current views on policy issues. On the flip side, a candidate is faced with the dilemma of never grasping the full range of factors that affect citizens’ vote decisions. Having rational, informed, voters who have well-articulated views on policy issues, is not sufficient for the candidate to ascertain how the votes will be cast. In addition, the data he possesses are likely to contain a large amount of error.

Some modelling studies have been conducted regarding the growth of political parties and voters. In most of the studies, statistical methodology is used by considering that two political parties are competing for the voters’ class [6]-[9]. In particular, All Progressives Congress (APC) and People Democratic Party (PDP) in Nigeria compete for the votes of undecided voters [10][11]. They have considered the role of election campaigns by all competing candidates on these voters. Khan describes a political party arena of the multiparty political system using nonlinear differential equations with time delay [10]. Their mathematical model gives a comprehensive overview of political competitors. Similarly, Huckfeldt and Kohfeld presents two mathematical models of political parties, one linear and another nonlinear regarding the electoral stability and the decline of class in democratic politics by considering two political parties [9]. In the linear model they have considered behavioural independence within the electorate and in the nonlinear model policy platform, ideology and candidate popularity have been captured. Their model’s analysis is a clear picture of politics today.

The current system of democratic election in Nigeria is totally paper-based and manual that takes a lot of time and huge financial expenses [12]. The voters are registered just before the poll so the election commission gets sometimes all the necessary arrangements within this short period of time. INEC added a new voters’ register with the previous voters so that the people who have deceased by this time may be considered as the existing voters if they are not
informed [13]. So, people may not bestow their faith on the voters’ list as it contains numerous fake voters. Again, the authority itself may be corrupted and can allow some fake voters to participate. Voters who reside in foreign countries or miss the registration processes are not considered voters until they inform the authorities, however, in this case, most people do not usually show any interest in the process [14]. Voters may also change place of residence in the period between two elections and if the authority is not informed, they are not considered as voters in that location though constitutionally, they are voters. Therefore, they miss the chance to express this civic right by casting their vote. Sometimes people ruin their votes by stamping on two or more signs mistakenly. This is also a drawback of the paper-based voting system. While casting votes, the acting officers present in the polling unit mark a voter with a black ink on his or her nail but it is removable. So, there is a chance for casting illegal votes. Again, the votes are counted manually, there is a probability that the process becomes a gradual fraught with inaccuracies. These problems necessitated thoughts towards inventing a new system that will reduce corruption and increase accuracy in electoral processes. The concept of the advanced electronic voting system comes from this necessity [15].

2. MATERIALS AND METHODS

2.1. The Proposed Model. In this chapter, we give a detail explanation of model formulation and present model equations. Equilibrium points and stability analysis of the model are discussed. We first present the following assumptions:

i. any opposition party can emerge to be the ruling political party
ii. movement of eligible voters from one political party to another is defection
iii. political party campaigners or godfathers of a given party are its members
iv. the model ignores votes buying, elections cancellation, declaration of election results inconclusive and antiparty activities.
v. candidate integrity, precedents and political ideology of a party can woo voters confidence
vi. the class of I, J, E, S and T are not members of any political party.

The influx of voters into the system is determined by the registration rate of voter cards \( \Gamma \). Hence, voters enter the system at rate \( \Gamma \) into the eligible voter class \( V \). Eligible voters listen to the campaign and are influenced by political party campaigners of a given party \( G_i \), for \( i = \)
1, 2, 3. Hence, the rate of political impulse to eligible voters per party is \( \phi \varphi P \), where, \( \phi \) is the measure of the campaign strategy success of a given political party; \( P \) and \( \varphi \) is the measure of the positive influence of political party campaigners on an eligible voter. Therefore, the waves of political influence on eligible voters is given by \( \phi \varphi P_1, \phi \varphi P_2, \) and \( \phi \varphi P_3 \) for the ruling party \( P_1 \), major opposition party \( P_2 \) and minority parties \( P_3 \), respectively. The choice an eligible voter makes is influenced by a voter interest in a given political party \( P \) and the positive influence of the political party campaigners of the party on the voter. The probability for an eligible voter to accept the ideology of a political party is given by \( \phi \varphi \left( \frac{V}{N} \right) \) where, \( N \) is the total population. Thus, an eligible voter enters a political party \( P \) at a per capita rate of \( \alpha \beta \) where, \( \alpha \) is interest rate of an eligible voter and \( \beta \) is per capita campaign rate of a given party member to an eligible voter that is convinced to join the party. Hence, the rate at which eligible voters become members of a given political party is given by \( \alpha \beta V \left( \frac{P}{N} \right) \) where, \( \frac{P}{N} \) is the probability of an eligible voter coming in contact with a member of a given political party \( P \). The model also accounts for the defection between political parties. The per capita defection rate from one political party to another is denoted by \( \gamma \). Thus, the rate at which registered voters defect from party \( P \) to \( P \) is given by \( \gamma \left( \frac{P}{N} \right) \) and since there is no defection in the same political party we have \( \begin{bmatrix} 0 & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & 0 & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & 0 \end{bmatrix} \) where, \( \gamma_{11} = \gamma_{22} = \gamma_{33} = 0 \).

The political consultancy rate \( h \) of a given party \( P \) to its political party campaigners \( G \) or the rate \( x \) at which members of a given party \( P \) become its political party campaigners \( G \) is per capita rate given by \( (h + x)P \). The model permutes per capital influenced rate \( q \) of a political party campaigner \( G \) on its party activities. Therefore, members of a given political party \( P \) are being influenced by their political party campaigners \( G \) at the rate \( q \gamma \left( \frac{P}{N} \right) \).

Modification parameter associated with reduced influence of a political party campaigner \( G \)
and how a political party campaigner developed weak political prowess in the system is also measured in per capita rate of \((1 - q_i)\tau_i\) where, \((1 - q_i)\) is the reduced influence of a political party campaigner \(G_i\) and \(\tau_i\) is a weak political prowess of such political party campaigner. Hence, the political party campaigners of a given party \(P_i\) become politically uninfluential in the space of time at a rate of \((1 - q_i)\tau_iG_i\).

In a given political party \(P_i\), we estimate the influences of a political party campaigner \(G_i\) to electoral body and thugs in proportions that is enhanced by information \(\omega_i\), money \(\theta_i\) and power \(\eta_i\). This is a common strategy of political permutation in order to win an election. All the political parties in the system seek to do this through their political party campaigners or political allies. A political campaigner influenced the electoral body and got connected to thugs at a per capita rate of \(\omega_i\theta_i\eta_i\). Therefore, the proportion of influence from political party campaigners of a given party \(P_i\) to electoral body \(I\), is given by \((\omega_i\theta_i\eta_i)G_i\), and to the thugs \(T\) is \((1 - \omega_i\theta_i\eta_i)G_i\). Public enlightenment information from electoral body on electoral procedures within the system reaches eligible voters \(V\), security personnel \(S\), electoral observers \(E\), and judiciary \(J\) at the rates of \(\psi_I, \psi_S, \psi_E,\) and \(\psi_J\), respectively. The thugs caused chaos during elections that are being fuelled by political influence in the proportions of \(\nu(1 - \rho)T\) to security personnel and the attack rate to the electoral body is \(\rho T\). We have assumed retirement rate of the political classes and per capita rate of felony conviction or death to be \((\mu + \sigma)\). This rate is applicable to \(V, P, G,\) and \(T\). The classes of \(I, J, E,\) and \(S\) also reduced through retirement or death rate \(\delta\). And the judiciary interprets electoral reforms to electoral body at rate of \(\psi_\delta\) and \(\varepsilon\) measured the reduction in compliance from thugs for causing chaos during elections. Figure 1 is a schematic representation of our proposed model.
Table 1. Model State Variables

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V(t)$</td>
<td>Number of eligible voters 18 years and above at time $t$</td>
</tr>
<tr>
<td>$I(t)$</td>
<td>Electoral body at time $t$</td>
</tr>
<tr>
<td>$P_1(t)$</td>
<td>Ruling political party at time $t$</td>
</tr>
<tr>
<td>$P_2(t)$</td>
<td>Major opposition political party at time $t$</td>
</tr>
<tr>
<td>$P_3(t)$</td>
<td>Minority political parties at time $t$</td>
</tr>
<tr>
<td>$G_1(t)$</td>
<td>Political party campaigners of the ruling party</td>
</tr>
<tr>
<td>$G_2(t)$</td>
<td>Political party campaigners of opposition party</td>
</tr>
<tr>
<td>$G_3(t)$</td>
<td>Political party campaigners of minority parties</td>
</tr>
<tr>
<td>$S(t)$</td>
<td>Number of security personnel deployed during election</td>
</tr>
<tr>
<td>$E(t)$</td>
<td>Number of electoral observers</td>
</tr>
<tr>
<td>$J(t)$</td>
<td>Judiciary/electoral tribunals</td>
</tr>
<tr>
<td>$T(t)$</td>
<td>Number of electoral thugs during election</td>
</tr>
</tbody>
</table>

The solid lines in the diagram represent the influx rates at which individuals enter and leave the system and the rates at which individuals initially choose their political parties. This initial choice depends on how any of the three political groups $P_1$, $P_2$ or $P_3$ convinced an eligible
voter. The dashed lines signified the political influence and information within the system.

Variables and parameters in the model are presented in Tables 1 and 2.

Table 2. Model Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$</td>
<td>Registration rate of voter cards</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>Interest rate of eligible voters to the ruling party</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>Interest rate of eligible voters to major opposition party</td>
</tr>
<tr>
<td>$\alpha_3$</td>
<td>Interest rate of eligible voters to minority parties</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>Campaign strategy of the ruling party to eligible voters</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>Campaign strategy of major opposition party to eligible voters</td>
</tr>
<tr>
<td>$\phi_3$</td>
<td>Campaign strategy of minority parties to eligible voters</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>Per capita campaign rate of the ruling party to eligible voters</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>Per capita campaign rate of major opposition party to eligible voters</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>Per capita campaign rate of minority parties to eligible voters</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>Influence of political party campaigners of the ruling party to voters</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>Influence of political party campaigners of major opposition party to voters</td>
</tr>
<tr>
<td>$\phi_3$</td>
<td>Influence of political party campaigners of minority parties to voters</td>
</tr>
<tr>
<td>$\gamma_{12}$</td>
<td>Defection rate of the ruling party to major opposition party</td>
</tr>
<tr>
<td>$\gamma_{21}$</td>
<td>Defection rate of major opposition party to the ruling party</td>
</tr>
<tr>
<td>$\gamma_{31}$</td>
<td>Defection rate of minority parties to the ruling parties</td>
</tr>
<tr>
<td>$\gamma_{13}$</td>
<td>Defection rate of the ruling party to minority parties</td>
</tr>
<tr>
<td>$\gamma_{23}$</td>
<td>Defection rate of major opposition party to minority parties</td>
</tr>
<tr>
<td>$\gamma_{32}$</td>
<td>Defection rate of minority parties to major opposition party</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Retirement rate of political classes which is applicable to $V, P_1, P_2, P_3, G_1, G_2, G_3, T.$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Per capita rate of felony conviction or death applicable to $V, P_1, P_2, P_3, G_1, G_2, G_3, T.$</td>
</tr>
<tr>
<td>$h_1$</td>
<td>Consultancy rate of the ruling party to its political party campaigners</td>
</tr>
<tr>
<td>$h_2$</td>
<td>Consultancy rate of major opposition party to its political party campaigners</td>
</tr>
<tr>
<td>$h_3$</td>
<td>Consultancy rate of minority parties to their political party campaigners</td>
</tr>
<tr>
<td>$x_1$</td>
<td>Rate at which members of the ruling party become its political party campaigners</td>
</tr>
</tbody>
</table>
Parameter | Description
---|---
$x_2$ | Rate at which members of major opposition party become its political party campaigners
$x_3$ | Rate at which members of minority parties become their political party campaigners
$q_1$ | Influence of the ruling party campaigners to its party activities
$q_2$ | Influence of major opposition party campaigners to its party activities
$q_3$ | Influence of minority parties campaigners to their party activities
$(1 - q_1)$ | Modification parameter associated with reduced influence of political party campaigners of the ruling party
$(1 - q_2)$ | Modification parameter associated with reduced influence of political party campaigners of major opposition party
$(1 - q_3)$ | Modification parameter associated with reduced influence of political party campaigners of minority parties
$r_1$ | Rate at which political party campaigners of the ruling party developed weak political prowess
$r_2$ | Rate at which political party campaigners of major opposition party developed weak political prowess
$r_3$ | Rate at which political party campaigners of minority parties developed weak political prowess
$\omega_1\theta_1\eta_1$ | Estimates influence of political party campaigners of the ruling party to electoral body and therefore, $(1 - \omega_1\theta_1\eta_1)$ goes to thugs which is both enhanced by information, money and power
$\omega_2\theta_2\eta_2$ | Estimates influence of political party campaigners of major opposition party to electoral body and therefore, $(1 - \omega_2\theta_2\eta_2)$ goes to thugs which is both enhanced by information, money and power
$\omega_3\theta_3\eta_3$ | Estimates influence of political party campaigners of minority parties to electoral body and therefore, $(1 - \omega_3\theta_3\eta_3)$ goes to thugs which is both enhanced by information, money and power
$\delta$ | Retirement or death of a member of constituted authorities which is applicable to $I$, $J$, $E$, and $S$
$\nu$ | Compromising rate of security personnel during elections
$(1 - \rho)$ | Proportion of thugs attacks to security personnel and therefore, $\rho$ is the proportion of thugs attacks to electoral body during election
$\psi_1$ | Flow of electoral information from electoral body to eligible voters
$\psi_2$ | Compliance rate of security information from electoral body to security personnel during election
$\psi_3$ | Flow of electoral information from electoral body to electoral observers
$\psi_4$ | Flow of electoral information from electoral body to electoral tribunals
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1 - p)$</td>
<td>Proportion of security information from electoral observers to security personnel and therefore, $p$ is the proportion of security information passed to electoral body during election</td>
</tr>
<tr>
<td>$\psi_s$</td>
<td>Modification parameter associated with interpretation of electoral reforms to electoral body by the electoral tribunals</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>Is measuring the reduction in compliance from thugs to cause chaos during elections</td>
</tr>
</tbody>
</table>

Based on the explanation in section 2.1, we proposed the following model equations:

$$
\frac{d\bar{V}}{dt} = \Gamma + \sum_{i=1}^{3} \phi_i \varphi_i \bar{P}_i - \sum_{i=1}^{3} \alpha_i \beta_i \bar{P}_i \frac{\bar{V}}{N} - (\mu + \delta)\bar{V} + \psi_1 \bar{I} (1)
$$

$$
\frac{d\bar{P}_1}{dt} = [\alpha_1 \beta_1 \bar{V} + (\gamma_21 - \gamma_{12}) \bar{P}_2 + (\gamma_{31} - \gamma_{13}) \bar{P}_3] \frac{\bar{P}_1}{N} - (h_1 + x_1 + \phi_1 \varphi_1 + \mu + \sigma)\bar{P}_1 + q_1 \bar{G}_1 (2)
$$

$$
\frac{d\bar{P}_2}{dt} = [\alpha_2 \beta_2 \bar{V} + (\gamma_{32} - \gamma_{23}) \bar{P}_3 + (\gamma_{12} - \gamma_{21}) \bar{P}_1] \frac{\bar{P}_2}{N} - (h_2 + x_2 + \phi_2 \varphi_2 + \mu + \sigma)\bar{P}_2 + q_2 \bar{G}_2 (3)
$$

$$
\frac{d\bar{P}_3}{dt} = [\alpha_3 \beta_3 \bar{V} + (\gamma_{23} - \gamma_{32}) \bar{P}_2 + (\gamma_{13} - \gamma_{31}) \bar{P}_1] \frac{\bar{P}_3}{N} - (h_3 + x_3 + \phi_3 \varphi_3 + \mu + \sigma)\bar{P}_3 + q_3 \bar{G}_3 (4)
$$

$$
\frac{d\bar{G}_1}{dt} = (h_1 + x_1)\bar{P}_1 + [q_1 + (1 - q_1)\tau_1 + \omega_1 \theta_1 \eta_1 + (1 - \omega_1 \theta_1 \eta_1) + \mu + \delta] \bar{G}_1 (5)
$$

$$
\frac{d\bar{G}_2}{dt} = (h_2 + x_2)\bar{P}_2 + [q_2 + (1 - q_2)\tau_2 + \omega_2 \theta_2 \eta_2 + (1 - \omega_2 \theta_2 \eta_2) + \mu + \delta] \bar{G}_2 (6)
$$

$$
\frac{d\bar{G}_3}{dt} = (h_3 + x_3)\bar{P}_3 + [q_3 + (1 - q_3)\tau_3 + \omega_3 \theta_3 \eta_3 + (1 - \omega_3 \theta_3 \eta_3) + \mu + \delta] \bar{G}_3 (7)
$$

$$
\frac{d\bar{I}}{dt} = \omega_1 \theta_1 \eta_1 \bar{G}_1 + \omega_2 \theta_2 \eta_2 \bar{G}_2 + \omega_3 \theta_3 \eta_3 \bar{G}_3 + p\bar{E} + \rho\bar{T} + \psi_5\bar{I} - (\psi_1 + \psi_2 + \psi_3 + \psi_4 + \delta)\bar{I} (8)
$$

$$
\frac{d\bar{T}}{dt} = (1 - \omega_1 \theta_1 \eta_1)\bar{G}_1 + (1 - \omega_2 \theta_2 \eta_2)\bar{G}_2 + (1 - \omega_3 \theta_3 \eta_3)\bar{G}_3 - (\rho + e_1 + \epsilon + \mu + \delta)\bar{T} (9)
$$

$$
\frac{d\bar{E}}{dt} = e_2 \bar{E} + \psi_2 \bar{I} + e_4 \bar{T} - \delta \bar{S} (10)
$$

$$
\frac{d\bar{S}}{dt} = \psi_3 \bar{l} - (p + e_2 + \delta)\bar{E} (11)
$$

$$
\frac{d\bar{F}}{dt} = \psi_4 \bar{I} - (\psi_5 + \delta)\bar{F} (12)
$$

$$\bar{N} = \bar{V} + \sum_{i=1}^{3} (\bar{P}_i + \bar{G}_i) + \bar{I} + \bar{T} + \bar{S} + \bar{E} + \bar{F} (13)$$
where \( V(0) > 0, A(0) > 0, B(0) > 0, C(0) > 0, G_i(0) > 0, G_2(0) > 0, G_3(0) > 0, I(0) > 0, \)

\( T(0) > 0, S(0) > 0, E(0) > 0, \) and \( J(0) > 0. \)

2.2. Analyses of the Election Model. For the purpose of convenience, we simplified model

equations (1) – (12) as follows:

\[
\frac{d\bar{V}}{dt} = \Gamma + \sum_{i=1}^{3} d_i \bar{P}_i - \left[ \sum_{i=1}^{3} \Omega_i \bar{P}_i \right] \frac{\bar{V}}{N} - b \bar{V} + \psi_1 \bar{I} \tag{14}
\]

\[
\frac{d\bar{P}_1}{dt} = [\Omega_1 \bar{V} - \gamma_1 \bar{P}_2 - \gamma_3 \bar{P}_3] \frac{\bar{P}_1}{N} - k_1 \bar{P}_1 + q_1 \bar{G}_1 \tag{15}
\]

\[
\frac{d\bar{P}_2}{dt} = [\Omega_2 \bar{V} - \gamma_2 \bar{P}_3 - \gamma_1 \bar{P}_1] \frac{\bar{P}_2}{N} - k_2 \bar{P}_2 + q_2 \bar{G}_2 \tag{16}
\]

\[
\frac{d\bar{P}_3}{dt} = [\Omega_3 \bar{V} - \gamma_2 \bar{P}_2 - \gamma_3 \bar{P}_1] \frac{\bar{P}_3}{N} - k_3 \bar{P}_3 + q_3 \bar{G}_3 \tag{17}
\]

\[
\frac{d\bar{G}_1}{dt} = r_1 \bar{P}_1 - k_4 \bar{G}_1 \tag{18}
\]

\[
\frac{d\bar{G}_2}{dt} = r_2 \bar{P}_2 - k_5 \bar{G}_2 \tag{19}
\]

\[
\frac{d\bar{G}_3}{dt} = r_3 \bar{P}_3 - k_6 \bar{G}_3 \tag{20}
\]

\[
\frac{d\bar{I}}{dt} = a_1 \bar{G}_1 + a_3 \bar{G}_2 + a_5 \bar{G}_3 + p \bar{E} + \rho \bar{T} + \psi_3 \bar{I} - k_7 \bar{I} \tag{21}
\]

\[
\frac{d\bar{T}}{dt} = a_2 \bar{G}_1 + a_4 \bar{G}_2 + a_6 \bar{G}_3 - k_8 \bar{T} \tag{22}
\]

\[
\frac{d\bar{S}}{dt} = e_2 \bar{E} + \psi_2 \bar{I} + e_1 \bar{T} - \delta \bar{S} \tag{23}
\]

\[
\frac{d\bar{E}}{dt} = \psi_3 \bar{I} - k_9 \bar{E} \tag{24}
\]

\[
\frac{d\bar{F}}{dt} = \psi_4 \bar{I} - k_{10} \bar{F} \tag{25}
\]

\( \frac{d}{dt} \) where \( \beta = \mu + \delta, \gamma_1 = \gamma_12 - \gamma_21, \gamma_2 = \gamma_23 - \gamma_32, \gamma_3 = \gamma_31 - \gamma_13; \ d_1 = \phi_1 \varphi_1; \ d_2 = \phi_2 \varphi_2; \ d_3 = \phi_3 \varphi_3; \ e_1 = \nu(1 - \rho); \ e_2 = 1 - p; \ a_1 = \omega_1 \theta_1 \eta_1; \ a_2 = 1 - a_1; \ a_3 = \omega_2 \theta_2 \eta_2; \)

\( a_4 = 1 - a_3; \ a_5 = \omega_3 \theta_3 \eta_3; \ a_6 = 1 - a_5; \ k_1 = r_1 + d_1 + b; \ k_2 = r_2 + d_2 + b; \ k_3 = r_3 + d_3 + b; \ k_4 = (q_1 + (1 - q_1)) r_1 + a_1 + a_2 + b; \ k_5 = (q_2 + (1 - q_2)) r_2 + a_3 + a_4 + b; \)

\( k_6 = (q_3 + (1 - q_3)) r_3 + a_5 + a_6 + b; \ k_7 = \psi_1 + \psi_2 + \psi_3 + \psi_4 + \delta; \ k_8 = \rho + e_1 + e + b; \ k_9 = p + e_2 + \delta; \ Omega_1 = \alpha_1 \beta_1; \ Omega_2 = \alpha_2 \beta_2; \ Omega_3 = \alpha_3 \beta_3; \ r_1 = h_1 + x_1; \ r_2 = h_2 + x_2; \ r_3 = h_3 + x_3; \ k_{10} = \psi_5 + \delta & i = 1, 2, 3 \)
2.3. Stability Analysis of the Election Model. In this subsection, we discussed the stability analysis of the election model by introducing the following new state variables:

\[ V = \frac{\bar{V}}{N}; \quad P_i = \frac{\bar{P}_i}{N}; \quad G_i = \frac{\bar{G}_i}{N}, \quad I = \frac{\bar{I}}{N}; \quad T = \frac{\bar{T}}{N}; \quad S = \frac{\bar{S}}{N}; \quad E = \frac{\bar{E}}{N}; \quad J = \frac{\bar{J}}{N}; \quad \text{and} \ N = \frac{\bar{b}N}{\bar{r}}. \]

to scale each subpopulation as a proportion of the total population \( N \). Hence,

\[ V + P_1 + P_2 + P_3 + sdG_1 + G_2 + G_3 + I + T + S + E + J = 1. \]

Models (1) to (12) become,

\[
\begin{align*}
\frac{dV}{dt} &= 1 + \sum_{i=1}^{3} d_i^* P_i + \psi_1^* I - \sum_{i=1}^{3} \Omega_i^* P_i V - V \frac{dN}{N} \frac{dN}{dt}, \\
\frac{dP_1}{dt} &= \left[ \Omega_1^* V - \gamma_1^* P_2 - \gamma_2^* P_3 \right] P_1 - k_1^* P_1 + q_1^* G_1 - \frac{P_1}{N} \frac{dN}{dt}, \\
\frac{dP_2}{dt} &= \left[ \Omega_2^* V - \gamma_2^* P_3 - \gamma_3^* P_1 \right] P_2 - k_2^* P_2 + q_2^* G_2 - \frac{P_2}{N} \frac{dN}{dt}, \\
\frac{dP_3}{dt} &= \left[ \Omega_3^* V - \gamma_3^* P_1 - \gamma_1^* P_3 \right] P_3 - k_3^* P_3 + q_3^* G_3 - \frac{P_3}{N} \frac{dN}{dt}, \\
\frac{dG_1}{dt} &= r_1^* P_1 - k_4^* G_1 - \frac{G_1}{N} \frac{dN}{dt}, \\
\frac{dG_2}{dt} &= r_2^* P_2 - k_6^* G_2 - \frac{G_2}{N} \frac{dN}{dt}, \\
\frac{dG_3}{dt} &= r_3^* P_3 - k_8^* G_3 - \frac{G_3}{N} \frac{dN}{dt}, \\
\frac{dI}{dt} &= a_1^* G_1 - a_2^* G_2 + a_5^* G_3 + P^* E + \rho^* T + \psi_5^* J - k_I^* I - \frac{I}{N} \frac{dN}{dt}, \\
\frac{dT}{dt} &= a_2^* G_1 - a_3^* G_2 + a_6^* G_3 - k_5^* T - \frac{T}{N} \frac{dN}{dt}, \\
\frac{dS}{dt} &= e_2^* E + \psi_2^* I + a_6^* G_3 - e_1^* T - \delta^* S - \frac{S}{N} \frac{dN}{dt}, \\
\frac{dE}{dt} &= \psi_3^* I - k_9^* E - \frac{E}{N} \frac{dN}{dt}, \\
\frac{dJ}{dt} &= \psi_4^* I - k_{10}^* J - \frac{J}{N} \frac{dN}{dt}, \\
\frac{dN}{dt} &= \left[ 1 + (\delta^* - 1)(1 + I) + S + E \right] + \sum_{i=1}^{3} (1 - q_i) \tau_i^* \bar{G}_i + \epsilon^* T \right] N. 
\end{align*}
\]

where \( t^* = bt; \quad k_i^* = \frac{k_i}{b} \).

For \( i = 1, \ldots, 10; \quad \tau_i^* = \frac{\tau_i}{b}; \quad d_i^* = \frac{d_i}{b}; \quad \Omega_i^* = \frac{\Omega_i}{b}. \)

For \( i = 1, \ldots, 3; \quad a_i^* = \frac{a_i}{b}; \quad \rho_i^* = \frac{\rho_i}{b}. \)

For \( i = 1, \ldots, 6; \quad e_i^* = \frac{e_i}{b}. \)

For \( i = 1, \ldots, 2; \quad \delta_i^* = \frac{\delta_i}{b} \) and \( \tau_i^* = \frac{\tau_i}{b}. \)

For \( i = 1, \ldots, 3 \) dropping the asterisk notation and since

\[ V = 1 - \sum_{i=1}^{3} (P_i + G_i) + I + T + S + E + J \]

\[
\begin{align*}
\frac{dP_1}{dt} &= \left[ \Omega_1 V - \gamma_1 P_2 - \gamma_3 P_3 \right] P_1 - k_1 P_1 + q_1 G_1 - \frac{P_1}{N} \frac{dN}{dt}, \\
\frac{dP_2}{dt} &= \left[ \Omega_2 V - \gamma_2 P_3 - \gamma_1 P_1 \right] P_2 - k_2 P_2 + q_2 G_2 - \frac{P_2}{N} \frac{dN}{dt}, \\
\frac{dP_3}{dt} &= \left[ \Omega_3 V - \gamma_3 P_1 - \gamma_2 P_3 \right] P_3 - k_3 P_3 + q_3 G_3 - \frac{P_3}{N} \frac{dN}{dt}. 
\end{align*}
\]
\[
\frac{dP_3}{dt} = [\Omega_3 V - \gamma_2 P_2 - \gamma_3 P_3] P_3 - k_3 P_3 + q_3 G_3 - \frac{P_3 dN}{N dt}
\] (42)

\[
\frac{dG_1}{dt} = r_1 P_1 - k_4 G_1 - \frac{G_1 dN}{N dt}
\] (43)

\[
\frac{dG_2}{dt} = r_2 P_2 - k_5 G_2 - \frac{G_2 dN}{N dt}
\] (44)

\[
\frac{dG_3}{dt} = r_3 P_3 - k_6 G_3 - \frac{G_3 dN}{N dt}
\] (45)

\[
\frac{dI}{dt} = r_3 G_1 - a_3 G_2 + a_5 G_3 + pE + \rho T + \psi_3 I - k_7 I - \frac{l dN}{N dt}
\] (46)

\[
\frac{dT}{dt} = r_2 G_1 - a_4 G_2 + a_6 G_3 - k_8 T - \frac{T dN}{N dt}
\] (47)

\[
\frac{dS}{dt} = e_2 E + \psi_2 I + a_6 G_3 - e_1 T - \delta S - \frac{S dN}{N dt}
\] (48)

\[
\frac{dE}{dt} = \psi_3 I - k_9 E - \frac{E dN}{N dt}
\] (49)

\[
\frac{dJ}{dt} = \psi_4 I - k_{10} - \frac{J dN}{N dt}
\] (50)

\[
\frac{dN}{dt} = 1 - \frac{1}{1 + (\delta - 1)(I + J + S + E) + \sum (1 - q_i) \tau_i G_i - \varepsilon T] N}{N dt}
\] (51)

### 2.4. Equilibrium Points of Election Model

In this subsection, we discussed the equilibrium points of election model. At the equilibrium points,

\[
\frac{dN}{dt} = \frac{dP_1}{dt} = \frac{dP_2}{dt} = \frac{dP_3}{dt} = \frac{dG_1}{dt} = \frac{dG_2}{dt} = \frac{dG_3}{dt} = \frac{dI}{dt} = \frac{dT}{dt} = \frac{dS}{dt} = \frac{dE}{dt} = 0
\]

Thus, equations (40) – (51) becomes

\[
[\Omega_1 V - \gamma_1 P_2 - \gamma_3 P_3] P_1 - k_1 P_1 + q_1 G_1 = 0
\] (52)

\[
[\Omega_2 V - \gamma_2 P_3 - \gamma_3 P_1] P_2 - k_2 P_2 + q_2 G_2 = 0
\] (53)

\[
[\Omega_3 V - \gamma_2 P_2 - \gamma_3 P_1] P_3 - k_3 P_3 + q_3 G_3 = 0
\] (54)

\[
r_1 P_1 - k_4 G_1 = 0
\] (55)

\[
r_2 P_2 - k_5 G_2 = 0
\] (56)

\[
r_3 P_3 - k_6 G_3 = 0
\] (57)

\[
a_4 G_1 - a_3 G_2 + a_5 G_3 + pE + \rho T + \psi_3 I - k_7 I = 0
\] (58)

\[
a_2 G_1 - a_4 G_2 + a_6 G_3 - k_8 T = 0
\] (59)

\[
e_2 E + \psi_2 I + a_6 G_3 - e_1 T - \delta S = 0
\] (60)

\[
\psi_3 I - k_9 E = 0
\] (61)

\[
\psi_4 I - k_{10} = 0
\] (62)

\[
1 - \frac{1}{1 + (\delta - 1)(I + J + S + E) + \sum (1 - q_i) \tau_i G_i - \varepsilon T] N = 0
\] (63)

Solve (55) – (62) in terms of \(P_1 P_2 P_3\) to get

\[
G_1 = \frac{r_1 P_1}{k_4}
\] (64)
\[ G_2 = \frac{r_2 p_2}{k_5} \]  
\[ G_3 = \frac{r_3 p_3}{k_6} \]  
\[ T = \frac{A_0 p_1 + B_0 p_2 + C_0 p_3}{x_0} \]  
\[ I = \frac{A_1 p_1 + B_1 p_2 + C_1 p_3}{x_1} \]  
\[ S = \frac{A_2 p_1 + B_2 p_2 + C_2 p_3}{x_1} \]  
\[ E = \frac{\psi_2 l}{k_9} \]  
\[ J = \frac{\psi_4 l}{k_{10}} \]  

where \( A_0 = r_1 a_2 k_5 k_6 \), \( B_0 = r_2 a_4 k_4 k_6 \), \( C_0 = r_3 a_6 k_6 k_5 \), \( A_1 = r_1 k_5 k_6 k_7 k_9 (a_1 k_8 + a_2) \), \( B_1 = r_2 k_4 k_5 k_7 k_9 (a_3 k_8 + a_4) \), \( C_1 = r_3 k_4 k_6 k_7 k_9 (a_5 k_8 + a_6) \), \( X_0 = k_4 k_5 k_6 k_9 X_1 = X_0 (k_7 k_9 - \rho \psi_3 - \psi_4 \psi_5) \), \( A_2 = A_1 x_0 (e_2 \psi_3 + \psi_1 k_9) \), \( A_0 x_1 e_1 k_9 \), \( B_2 = B_1 x_0 (e_2 \psi_3 + \psi_2 k_9) + B_0 X_1 e_1 k_9 \), \( X_2 = X_0 X_1 k_9 \delta \), \( V = 1 - Z_1 P_1 - Z_2 P_2 - Z_3 P_3 \)

Substitute (64) – (66) in to (39) to have

\[ (72) \]

where

\[ A_3 = k_4 k_5 k_6 x_0 x_2 (k_6 k_9 + k_9 \psi_4 + k_{10} \psi_3) A_1 + k_5 k_6 k_9 k_{10} X_1 (A_0 X_2 + A_2 X_0) \]
\[ + k_5 k_6 k_9 k_{10} X_0 X_1 x_2 (k_4 + r_1), \]
\[ B_3 = k_4 k_5 k_6 x_0 x_2 (k_6 k_9 + k_9 \psi_4 + k_{10} \psi_3) B_1 + k_5 k_6 k_9 k_{10} X_1 (B_0 X_2 + B_2 X_0) \]
\[ + k_4 k_6 k_9 k_{10} X_0 X_1 x_2 (k_5 + r_2), \]
\[ C_3 = k_4 k_5 k_6 x_0 x_2 (k_6 k_9 + k_9 \psi_4 + k_{10} \psi_3) C_1 + k_5 k_6 k_9 k_{10} X_1 (C_0 X_2 + C_2 X_0) \]
\[ + k_4 k_5 k_9 k_{10} X_0 X_1 x_2 (k_6 + r_3), \]
\[ Z_1 = \frac{A_3 x_3}{X_3}, Z_2 = \frac{B_3 x_3}{X_3}, Z_3 = \frac{C_3 x_3}{X_3} \]

Substitute (64) – (66) and (72) in to (52) – (54) to obtain

\[ P_1 [P_1 (1 - Z_1 P_1 - Z_2 P_2 - Z_3 P_3) - \gamma_1 P_2 - \gamma_3 P_3 - k_1 + R_1] = 0 \]  
\[ P_2 [P_2 (1 - Z_1 P_1 - Z_2 P_2 - Z_3 P_3) - \gamma_2 P_3 + \gamma_1 P_1 - k_2 + R_2] = 0 \]  
\[ P_3 [P_3 (1 - Z_1 P_1 - Z_2 P_2 - Z_3 P_3) + \gamma_2 P_2 + \gamma_3 P_1 - k_3 + R_3] = 0 \]

Respectively such that
\[ R_1 = \frac{q_1 r_1}{k_4} R_2 = \frac{q_2 r_2}{k_4} \quad \text{and} \quad R_3 = \frac{q_3 r_3}{k_6} \]

The solution of (73) to (75) will give eight equilibrium points but due to the following facts that:

i. \( P_2 \) exist (that is \( P_2 \neq 0 \)) only when \( P_1 \) exist (that is \( P_1 \neq 0 \))

ii. \( P_3 \) exist (that is \( P_3 \neq 0 \)) only when \( P_2 \) exist (that is \( P_2 \neq 0 \))

Thus, is logically reasonable to consider the following equilibrium points.

a. **Party-free equilibrium**: Let \( Q_0 \) denote the party free equilibrium which corresponds to the absence of political parties. This means that \( Q_0 \) should satisfy

\[ P_1^0 = P_2^0 = P_3^0 = 0 \]

The solution of (73) - (75) gives \( P_1^0 = P_2^0 = P_3^0 = 0 \) which is then substituted to (63) to validate (76). Hence \( Q_0 \) exist unconditionally and nonpartisan system.

b. **Single party equilibrium**: The single party system is the system that legally recognizes the ruling party. This means that only the ruling party is permitted to exist \( (P_1^* \neq 0) \).

Let \( Q_1 \) denotes the equilibrium in a dictatorship system. Thus \( P_1^* > 0 \) such that \( P_2^* = P_3^* = 0 \) (79) becomes

\[ P_1[\Omega_1(1 - Z_1 P_1) - k_1 + R_1] = 0 \]  

which implies

\[ P_1^* = \frac{\Omega_1 + R_1 - k_1}{\Omega_1 Z_1} \]  

The single party equilibrium exists if any of the conditions below is satisfied

i. \( \Omega_1 + R_1 > k_1 \)

ii. \( \Omega_1 > k_1 \)

iii. \( R_1 > k_1 \)

c. **Dual party system.** For model (1) – (12), the dual party system exist in the absence of the minority party that is \( (P_3^* = 0) \). It means that the State officially permits only two political parties that is \( P_1^* \neq 0 \) and \( P_2^* \neq 0 \)

Let \( Q_2 \) denotes the dual party equilibrium such that (73) to (74) becomes

\[ \Omega_1(1 - Z_1 P_1^* - Z_2 P_2^*) - \gamma_1 P_2^* - k_1 + R_1 = 0 \]  

\[ \Omega_2(1 - Z_1 P_1^* - Z_2 P_2^*) - \gamma_1 P_2^* - k_2 + R_2 = 0 \]

Solve (79) and (80) simultaneously to obtain

\[ P_1^* = \frac{\Omega_1 Z_2 + \gamma_1 k_2 - \Omega_2 Z_1}{\gamma_1 (\gamma_1 + \Omega_1 Z_2 - \Omega_2 Z_1)} \]  

\[ P_2^* = \frac{\Omega_1 Z_2 + \gamma_1 (k_2 - R_2) - (k_1 - R_1) \gamma_1 Z_1}{\gamma_1 (\gamma_1 + \Omega_1 Z_2 - \Omega_2 Z_1)} \]
respectively. Hence, the dual party equilibrium that is $P_1^* > 0$, $P_2^* > 0$ and $P_3^* = 0$ exist provided the model’s parameters satisfy the following conditions

$$\Omega_1Z_2 > \Omega_2Z_1,$$ \hspace{1cm} (83)

$$\Omega_1Z_2 + \gamma_1(k_2 - R_2) - \Omega_2[\gamma_1 + Z_2(k_1 - R_1)] > 0$$ \hspace{1cm} (84)

$$\Omega_1[\gamma_1 + Z_1(k_2 - R_2)] - (k_1 - R_1)(\gamma_1 - \Omega_2Z_1) > 0$$ \hspace{1cm} (85)

The components of $Q_2$ are easily gotten by substituting (87) and (88) and $P_2^* = 0$ in to (86)-(77)

**d. Three-party Equilibrium**

The coexistence of the ruling, opposition and minority political parties that is $(P_1^+ \neq 0, P_1^+ \neq 0$ and $P_1^+ \neq 0$) implies that the three-party exist. This further means that (73) – (75) becomes

$$\Omega_1(1 - Z_1P_1^+ - Z_2P_2^+ - Z_3P_3^+) - \gamma_3P_3^+ - \gamma_1P_2^+ - k_1 + R_1 = 0$$ \hspace{1cm} (87)

$$\Omega_2(1 - Z_1P_1^+ - Z_2P_2^+ - Z_3P_3^+) - \gamma_2P_2^+ - \gamma_1P_1^+ - k_2 + R_2 = 0$$ \hspace{1cm} (88)

$$\Omega_3(1 - Z_1P_1^+ - Z_2P_2^+ - Z_3P_3^+) + \gamma_3P_3^+ + \gamma_2P_2 - k_3 + R_3 = 0$$ \hspace{1cm} (89)

Solution of (87) – (89) gives

$$P_1^+ = \frac{\gamma_1 - \gamma_2}{\gamma_3}$$ \hspace{1cm} (90)

$$P_2^+ = \frac{\gamma_4 - \gamma_5}{\gamma_3}$$ \hspace{1cm} (91)

$$P_3^+ = \frac{\gamma_6 - \gamma_7}{\gamma_3}$$ \hspace{1cm} (92)

where $f_1 = \Omega_1Z_1$; $f_2 = \gamma_1 + \Omega_1Z_2$; $f_3 = \gamma_2 + \Omega_1Z_3Z_4 = k_1 - \Omega_1 - R_1$; $f_4 = \gamma_1 - \Omega_2Z_1$; $f_5 = \Omega_2Z_2$; and $f_6 = \gamma_2 + \Omega_2Z_3Z_5 = k_2 - \Omega_2 - R_2$

$$y_1 = f_2f_6Z_6 + f_5f_9Z_4 + f_8f_3Z_5y_2 = f_2f_9Z_5 + f_5f_3Z_6 + f_6f_6Z_4$$

$$y_3 = f_7f_2Z_6 - [f_1(f_5f_9 + f_6f_6) + f_4(f_2f_9 + f_8f_3) + f_7f_2f_3],$$

$$f_7 = y_3 - \Omega_3Z_1f_8 = y_2 - \Omega_3Z_2f_8 = \Omega_3Z_3$$

$$y_4 = f_1f_5Z_5 + f_4f_5Z_4 + f_7f_3Z_4y_5 = f_1f_6Z_6 + f_7f_4Z_4$$ and $y_7 = f_2f_7Z_5$

Therefore, the three-party equilibrium exists if the parameters of the model satisfy the following condition

i. $y_3 > 0$

ii. $y_4 > y_5$

iii. $y_6 > y_7$

The other components of the three-party equilibrium are easily gotten by substituting (90)-(92) into (64)-(71).
2.5. Local Stability of Party-free Equilibrium. In this subsection, we discussed the local stability of party-free equilibrium using the Jacobian matrix on the model equations (1) to (13).

The characteristic equation of the above is simplified with the aid of Maple 18 software as

\[(\lambda + 1)(\lambda + \delta)(\lambda + k_8)W_1W_2W_3W_4 = 0\]  \hspace{1cm} \text{(93)}

where

\[W_1 = \lambda^2 + (k_1 + k_4 - \Omega_1)\lambda + k_1k_4 - \Omega_1k_4 - q_1r_1\]  \hspace{1cm} \text{(94)}

\[W_2 = \lambda^2 + (k_2 + k_5 - \Omega_2)\lambda + k_2k_5 - \Omega_2k_5 - q_2r_2\]  \hspace{1cm} \text{(95)}

\[W_3 = \lambda^2 + (k_3 + k_6 - \Omega_3)\lambda + k_3k_6 - \Omega_3k_6 - q_3r_3\]  \hspace{1cm} \text{(96)}

\[W_4 = \lambda^3 + (k_7 + k_9 - k_{10})\lambda^2 + (k_7k_9 + k_7k_{10} + k_9k_{10} - \psi_4\psi_5 - \rho\psi_3)\lambda + k_7k_9k_{10} - k_9\psi_4\psi_5 - k_{10}\psi_3\rho\]  \hspace{1cm} \text{(97)}

It obvious from (93) that \(\lambda_1 = -1 < 0, \lambda_2 = -\delta < 0, \lambda_3 = -k_8 < 0\)

Using Routh-Hurwitz criterion for the characteristic equation of the form

\[\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0\]  \hspace{1cm} \text{(98)}

where

\(\lambda\) is the eigenvalue of (98) and \(n\) is the degree of (98)

\(a_i\) is the coefficient of (98) such that \(0 = i < n - 1\)

Then, (98) is locally asymptotically stable for

i. \(n = 2\) if and only if \(a_i > 0\)
Thus, (94) is stable if \( k_1 > \Omega_1 \) and \( k_4(k_1 - \Omega_1) - q_1 r_1 > 0 \)

Similarly, (95) – (96) are stable if \( k_2 > \Omega_2, k_5(k_2 - \Omega_2) - q_2 r_2 > 0, k_3 > \Omega_3 \) and

Finally, \( W_4 \) is stable since all its coefficients are positive.

In summary, the party-free equilibrium is locally asymptotically stable if and only if the following conditions are satisfied, otherwise unstable.

i. \( k_1 > \Omega_1, k_4(k_1 - \Omega_1) - q_1 r_1 > 0 \)

ii. \( k_2 > \Omega_2, k_5(k_2 - \Omega_2) - q_2 r_2 > 0, k_3 > \Omega_3 \)

iii. \( k_3 > \Omega_3, k_6(k_3 - \Omega_3) - q_3 r_3 > 0 \)

2.6. Boundedness of the Election Model. In this subsection, we discussed the boundedness of the election model and show the feasible region of the model. We begin by differentiating (13) with respect to time \( t \), to have

\[
\frac{d\bar{N}}{dt} = \frac{d\bar{V}}{dt} + \sum_{i=1}^{3} \frac{d(\bar{P}_i + \bar{G}_i)}{dt} + \frac{d\bar{S}}{dt} + \frac{d\bar{E}}{dt} + \frac{d\bar{J}}{dt} \tag{99}
\]

Substitute (14) – (25) in to (99) with further simplification to get

\[
\frac{d\bar{N}}{dt} = \Gamma - b[\bar{V} + \sum_{i=1}^{3}(\bar{P}_i + \bar{G}_i) + \bar{T}] - \varepsilon \bar{T} - \sum_{i=1}^{3}(1 - q_i)\tau_i \bar{g}_i - \delta(\bar{I} + \bar{S} + \bar{E} + \bar{J}) \tag{100}
\]

It is obvious from (13) to note

\[
\bar{N} - \bar{I} + \bar{S} + \bar{E} + \bar{J} = \bar{V} + \sum_{i=1}^{3}(\bar{P}_i + \bar{G}_i) + \bar{T} \tag{101}
\]

Thus, (100) becomes

\[
\frac{d\bar{N}}{dt} = \Gamma - b\bar{N} + (b - \delta)(\bar{I} + \bar{S} + \bar{E} + \bar{J}) - \sum_{i=1}^{3}(1 - q_i)\tau_i \bar{g}_i - \varepsilon \bar{T} \tag{102}
\]

Lemma 1: The feasibility region \( D \) defined by \( D = \{(V, P_1, P_2, P_3, G_1, G_2, G_3, I, T, S, E, J)eR_{+}^{12}, N \leq \frac{r}{b}\} \), with non-negative initial condition are positive invariant for model (1) to (12) if \( \delta > b \)

Proof

From (102)

\[
\frac{d\bar{N}}{dt} \leq \Gamma - b\bar{N} \text{ If } \delta > b \tag{103}
\]

Applying Birkhoff’s and Rota’s theorem on (30) to get

\[
N(t) \leq \frac{r}{b} + e^{-bt}(N_0 - \frac{r}{b}) \tag{104}
\]
As \( t \to \infty \), (104) becomes \( 0 \leq N(t) \leq \frac{r}{b} \). This shows that the model can be studied in the feasible region,

\[
D = \{(V, P_1, P_2, P_3, G_1, G_2, G_3, I, T, S, E, J) \in \mathbb{R}_+^{12} \mid N \leq \frac{r}{b}\}.
\]

It can be verified that \( D \) is positively invariant with respect to the model equations (1) – (12).

Table 3. The following model parameters serve as inputs into election model

<table>
<thead>
<tr>
<th>Parameters ( \Gamma )</th>
<th>Values</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_1 )</td>
<td>0.7</td>
<td>Plausible value</td>
</tr>
<tr>
<td>( \alpha_2 )</td>
<td>0.5</td>
<td>Plausible value</td>
</tr>
<tr>
<td>( \alpha_3 )</td>
<td>0.2</td>
<td>Plausible value</td>
</tr>
<tr>
<td>( \phi_1 )</td>
<td>0.3</td>
<td>estimated</td>
</tr>
<tr>
<td>( \phi_2 )</td>
<td>0.2</td>
<td>estimated</td>
</tr>
<tr>
<td>( \phi_3 )</td>
<td>0.1</td>
<td>estimated</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>0.9</td>
<td>estimated</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>0.7</td>
<td>estimated</td>
</tr>
<tr>
<td>( \beta_3 )</td>
<td>0.5</td>
<td>estimated</td>
</tr>
<tr>
<td>( \varphi_1 )</td>
<td>0.3</td>
<td>estimated</td>
</tr>
<tr>
<td>( \varphi_2 )</td>
<td>0.2</td>
<td>estimated</td>
</tr>
<tr>
<td>( \varphi_3 )</td>
<td>0.1</td>
<td>estimated</td>
</tr>
<tr>
<td>( \gamma_{12} )</td>
<td>0.1</td>
<td>estimated</td>
</tr>
<tr>
<td>( \gamma_{21} )</td>
<td>0.4</td>
<td>estimated</td>
</tr>
<tr>
<td>( \gamma_{31} )</td>
<td>0.6</td>
<td>estimated</td>
</tr>
<tr>
<td>( \gamma_{13} )</td>
<td>0.1</td>
<td>estimated</td>
</tr>
<tr>
<td>( \gamma_{23} )</td>
<td>0.1</td>
<td>estimated</td>
</tr>
<tr>
<td>( \gamma_{32} )</td>
<td>0.2</td>
<td>estimated</td>
</tr>
<tr>
<td>( \mu )</td>
<td>0.2</td>
<td>estimated</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>0.1</td>
<td>estimated</td>
</tr>
<tr>
<td>( h_1 )</td>
<td>0.8</td>
<td>estimated</td>
</tr>
<tr>
<td>( h_2 )</td>
<td>0.5</td>
<td>estimated</td>
</tr>
<tr>
<td>( h_3 )</td>
<td>0.1</td>
<td>estimated</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0.5</td>
<td>estimated</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0.5</td>
<td>estimated</td>
</tr>
</tbody>
</table>

3. RESULTS AND DISCUSSIONS

3.1. Sensitivity Analysis, Numerical Simulations and Curve fittings. In this section, we present results for research findings and discussions as follows: analysis of Presidential, Senate Seats, House of Representatives, and Governorship elections over the years. We first invoked
the application of curve fitting and numerical simulation to examine the relationship between one or more predictors to a response variable, with the goal of defining a "best fit" relationship within the model variables and parameters. A linear model of poly degree three is used to checkmate the trajectories of each political party.

3.2. Case 1-Analysis of Presidential Elections over the Years.

Table 4. Results of Presidential Election 2007-2019 (Number of Votes Obtained by Candidates)

<table>
<thead>
<tr>
<th>Status</th>
<th>2007</th>
<th>2011</th>
<th>2015</th>
<th>2019</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ruling Party</td>
<td>24638063</td>
<td>22495187</td>
<td>12853462</td>
<td>11262978</td>
<td>INEC</td>
</tr>
<tr>
<td>Major opposition party</td>
<td>6605299</td>
<td>12214853</td>
<td>15424921</td>
<td>15191847</td>
<td>INEC</td>
</tr>
<tr>
<td>Minority opposition parties</td>
<td>4045622</td>
<td>3499938</td>
<td>309481</td>
<td>869758</td>
<td>INEC</td>
</tr>
</tbody>
</table>

Table 5. Results of Presidential Election 2007-2019 by Percentage of Valid Votes

<table>
<thead>
<tr>
<th>Status</th>
<th>2007</th>
<th>2011</th>
<th>2015</th>
<th>2019</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ruling Party</td>
<td>69.60%</td>
<td>58.87%</td>
<td>44.96%</td>
<td>41.22%</td>
<td>INEC</td>
</tr>
<tr>
<td>Major opposition party</td>
<td>18.66%</td>
<td>31.97%</td>
<td>53.96%</td>
<td>55.60%</td>
<td>INEC</td>
</tr>
<tr>
<td>Minority opposition parties</td>
<td>11.74%</td>
<td>9.16%</td>
<td>1.08%</td>
<td>3.18%</td>
<td>INEC</td>
</tr>
</tbody>
</table>

Table 6. Results of Presidential Election 2007-2019

<table>
<thead>
<tr>
<th>Status</th>
<th>2007</th>
<th>2011</th>
<th>2015</th>
<th>2019</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ruling Party</td>
<td>33</td>
<td>25</td>
<td>16</td>
<td>18</td>
<td>INEC</td>
</tr>
<tr>
<td>Major opposition party</td>
<td>4</td>
<td>11</td>
<td>21</td>
<td>19</td>
<td>INEC</td>
</tr>
<tr>
<td>Minority opposition parties</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>INEC</td>
</tr>
</tbody>
</table>

Figure 2. Performance plots of parties in presidential elections.
Figures 2 and 3 show that the number of votes obtained by the ruling party has been declining over the years, as indicated by the downward trend in the curve of Figure 2. The ruling party obtained the highest number of votes in the 2007 election, with over 24 million votes, but has been obtaining fewer votes in each subsequent election. The major opposition party, on the other hand, has been gaining ground over the years, as indicated by the upward trend in the curve. The major opposition party obtained the lowest number of votes in the 2007 election, with just over 11 million votes, but has been obtaining more votes in each subsequent election. Minority opposition parties have not been able to pose a significant challenge to the ruling party and the major opposition party, as indicated by the relatively flat curve.

Figure 3. Bar chart representing political parties in presidential elections.
Figure 4. Fitted Curve for Major Opposition Party in Presidential Elections.

Figure 4 indicates that the performance of the major opposition party has been increasing steadily over the years, as indicated by the upward trend in the curve. The curve suggests that the major opposition party’s performance was at its lowest in the 2007 election and has been increasing steadily since then.

Figure 5. Fitted Curve for the Ruling Party in Presidential Elections

Figure 5 indicates that the performance of the ruling party has been declining over the years, as indicated by the downward trend in the curve. The curve suggests that the ruling party’s
performance peaked in the 2007 election and has been declining steadily. Curve fitting indicates that the performance of the ruling party is influenced by several factors, including the popularity of the ruling party’s policies, the performance of the economy, and the level of public trust in the ruling party.

Figure 6. Fitted Curve for Minor Opposition Parties in Presidential Elections

Figure 6 shows the performance of minority opposition parties, is relatively low and inconsistent over the years, as indicated by the flat curve. The linear model used to fit the curve indicates that the performance of minority opposition parties is influenced by several factors, including the popularity of their policies, the level of public trust in them, and their ability to form strong alliances within the structure of the parties.

3.3. Case 2-Analysis of Senate Seats Won at the National Assembly over the Years.

Table 7. Table of Results for Number of Seats won at the Senate by Political Parties

<table>
<thead>
<tr>
<th>Status</th>
<th>2007</th>
<th>2011</th>
<th>2015</th>
<th>2019</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ruling Party</td>
<td>85</td>
<td>71</td>
<td>48</td>
<td>45</td>
<td>INEC</td>
</tr>
<tr>
<td>Major opposition party</td>
<td>16</td>
<td>18</td>
<td>60</td>
<td>63</td>
<td>INEC</td>
</tr>
<tr>
<td>Minority opposition parties</td>
<td>8</td>
<td>20</td>
<td>1</td>
<td>1</td>
<td>INEC</td>
</tr>
</tbody>
</table>
Figures 7 and 8 are visual representations of the performance of the ruling party, the major opposition party, and minority opposition parties in senate elections over the years. The performance of the ruling party has been fluctuating over the years, with some years having lower performance than others. For example, the ruling party won 85 seats in 2007, but only 45 seats in 2019. This suggests that the ruling party has faced challenges in maintaining its dominance at the Senate, and that its performance has been affected by various factors such as changes in leadership, policy decisions, and public opinion.
Figure 9. Fitted Curve for the performance of the Ruling Party in Senate Elections

Overall, the graph of Figure 9 shows that the performance of the ruling party at the Senate has been fluctuating over the years, with some years having higher performance than others. The fitted curve provides a good representation of the trend in the performance of the ruling party, with a high goodness of fit (R-square of 1).

Figure 10. Fitted Curve for the performance of the Main Opposition Party in Senate Elections

The performance of the major opposition party at the Senate is depicted in Figure 10. The graph shows a fitted curve that represents a linear model similar to the performance of the ruling party, with coefficients p1, p2, p3, and p4. The coefficients are -28.33, 0.4167, 58.47, and 38.94, respectively. The goodness of fit is also provided, with SSE of 1.551e-24 and R-square of 1. This implies that the predictions perfectly fit the data. The curve starts at a high point in 2007, with the major opposition party having a high number of seats in the Senate. However, the performance of the major opposition party declined significantly in 2008, with
the number of seats decreasing sharply. The performance continued to decline in 2011, with the number of seats reaching a low point.

Figure 11. Fitted Curve for the performance of the Other Parties in Senate Elections

Figure 11 shows that the performance of minority opposition parties at the Senate has been fluctuating over the years. For instance, in 2019, the ruling party won 45 seats, the major opposition party won 63 seats, while minority opposition parties won only 1 seat. This suggests that the political landscape is dominated by the ruling party and the major opposition party, with minority opposition parties having limited influence. However, the performance of minority opposition parties can still be important in shaping the political discourse and in providing alternative viewpoints.

3.4. Case 3-Analysis for Seats won by House of Representatives (HoR) over the Years.

Table 8. Table of Results for Number of Seats won by Political Parties

<table>
<thead>
<tr>
<th>Status</th>
<th>2007</th>
<th>2011</th>
<th>2015</th>
<th>2019</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ruling Party</td>
<td>262</td>
<td>203</td>
<td>212</td>
<td>128</td>
<td>INEC</td>
</tr>
<tr>
<td>Major opposition party</td>
<td>62</td>
<td>69</td>
<td>140</td>
<td>202</td>
<td>INEC</td>
</tr>
<tr>
<td>Minority opposition parties</td>
<td>36</td>
<td>88</td>
<td>8</td>
<td>30</td>
<td>INEC</td>
</tr>
</tbody>
</table>
Figures 12 and 13 show that apart from the ruling party, the major opposition party has been the second-largest party at the HoR, with the number of seats won increasing over the years. The major opposition party won 62 seats in 2007, but 202 seats in 2019. This suggests that the major opposition party has been gaining support and influence over the years, and that it has been successful in challenging the dominance of the ruling party. Furthermore, minority opposition parties have had limited influence at the HoR, with the number of seats won fluctuating over the years. Minority opposition parties won 36 seats in 2007, but only 30 seats in 2019. This suggests that minority opposition parties have faced challenges in gaining support and influence, and that they have been overshadowed by the dominance of the ruling party and the growing influence of the major opposition party.
Figure 14. Fitted Curve for Ruling Party in the HoR Elections.

Figure 14 represents the performance of the ruling party in the HoR which provides a useful tool for analysing the trend in the number of seats won by the ruling party over the years. The curve shows that the performance of the ruling party has been relatively stable, with the number of seats won fluctuating within a narrow range.

Figure 15. Fitted Curve for Major Opposition Party in the HoR Elections

Figure 15 shows that the major opposition party has been successful in challenging the dominance of the ruling party at the HoR. The increase in the number of seats won by the major opposition party suggests that it has been gaining support and influence over the years. However, the linear model suggests that the performance of the major opposition party is
affected by various factors such as changes in leadership, policy decisions, and public opinion. The linear model can be useful for predicting future trends in the performance of the major opposition party and for informing policy decisions.

Figure 16. Fitted Curve for Minority Opposition Parties in the HoR Elections.

Figure 16 represents the performance of minority opposition parties in the HoR. The curve provides valuable insights into the political landscape of the country. The increasing trend in the number of seats won by minority opposition parties from 2007 to 2011 indicates a growing opposition to the ruling party. However, the sharp decline in the number of seats won by minority opposition parties in 2015 suggests that the ruling party regained its dominance in the HoR. The slight increase in the number of seats won by minority opposition parties in 2019 indicates that the opposition is still present, but it is not strong enough to challenge the ruling party's dominance.

3.5. Case 4: Analysis of Governorship Elections over the Years.

Table 9. Table of Results for Number of State won by Political Parties

<table>
<thead>
<tr>
<th>Status</th>
<th>2007</th>
<th>2011</th>
<th>2015</th>
<th>2019</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ruling Party</td>
<td>24</td>
<td>23</td>
<td>10</td>
<td>13</td>
<td>INEC</td>
</tr>
<tr>
<td>Major opposition party</td>
<td>5</td>
<td>6</td>
<td>21</td>
<td>20</td>
<td>INEC</td>
</tr>
<tr>
<td>Minority opposition parties</td>
<td>7</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>INEC</td>
</tr>
</tbody>
</table>
Figure 17. Performance plots of parties in governorship elections.

Figure 17 reflects the performance of the ruling party, major opposition party, and minority opposition parties in governorship elections from 2007 to 2019.

Figure 18. Bar chart for political parties in Governorship elections

The bar chart in Figure 18 compares the performance of political parties in different elections. The chart clearly shows the changes in the number of states won by each party over time. The chart also shows that other parties have not been successful in winning states and may struggle to do so in future elections.
Importantly, the curve of Figure 19 shows that the ruling party's performance has been inconsistent, with a gradual increase in the number of states won from 2007 to 2008, followed by a sharp drop in 2011, and a gradual increase from 2011 to 2015. The curve also shows a sharp drop in the number of states won by the ruling party in 2015, followed by a slight increase in 2019.

In comparison to the fitted curve for the ruling party, the fitted curve for the major opposition party in Figure 20 shows a different trend. While the ruling party's performance has been inconsistent, the major opposition party's performance has been consistent, with a gradual increase in the number of states won over the years. This suggests that the major opposition
party may be gaining ground in Nigerian politics and may be a viable alternative to the ruling party in future elections.

Figure 21. Fitted curve for the number of states won by the Minor Opposition Parties

Figure 21 shows that the performance of minority opposition parties has been declining over the years and may continue to do so in future elections. However, the slight increase in the number of states won by minority opposition parties in 2015 suggests that minority opposition parties may still be able to bounce back in Nigerian politics.

3.6. Suggestion for Further Studies. Based on our model analysis and findings in this study, we suggest that further studies on this topic should incorporate the following recommendations:

a. elections malpractices like result manipulations, unfair electoral processes, vote buying, snatching of ballot boxes, unwholesome intimidation of some eligible voters, cancellation of election results, and lack of INEC transparency in the conduct of elections.

b. use of religious and tribal sentiment to woo eligible voters.

c. use of government machineries to frustrate opposition parties by the government in power use of judiciary and money inducement to truncate political mandates by political big wigs.

4. CONCLUSIONS

In this study, electoral processes are being modelled using a schematic diagram and set of non-linear differential equations that describe the dynamics of elections. The model has been
used to predict possible patterns for electoral outcomes over the years for both the ruling parties, major opposition parties and minority opposition parties. Numerical simulation and curve fittings are used to identify sensitive parameters and data visualization. For each of the case studies, the fitted curve for the ruling party and the main opposition party shows a different trend. While the ruling party's performance has been inconsistent, the main opposition party's performance has been consistent, with a gradual increase in the number of states won over the years. This suggests that the main opposition party may be gaining ground in Nigerian politics and may be a viable alternative to the ruling party and vice versa to minority opposition parties in future elections. We are able to establish the fact that the outcome of an election is a function of voter’s trust, candidate’s popularity and policies of a political party. The model also proves that political party campaigners do not have long-term effect on deciding who wins in an election contest as their influence becomes weak and wane over time. Winning therefore becomes the decision of eligible voters and candidate’s credibility. The choice an eligible voter makes is influenced by a number of factors, including membership of a party and the interest that the voter has developed in such party’s ideologies. In addition, since democratic leadership is being subjected to electoral processes, voter’s decision at the polling units becomes fundamental in predicting the outcome of elections. The reflection of this reality is that; Electoral Reform Act as amended, must be fully implemented. The use of political thugs and other electoral malpractices to win elections should be highly discouraged in order to avoid political agitations, insecurity in the country, lack of positive representatives at the parliaments and the economic hardship posed by such leadership. This menace in the lack of free and fair elections could be brought under control or eliminated by using the result of this research.

REFERENCES


