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Journal of Multidisciplinary Applied Natural Science Vol. 3 No. 2 (2023) https://doi.org/10.47352/jmans.2774-3047.176
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Research Article

An Analytic Solution to The Inhomogeneous Verhulst Equation Using Multiple Expansion Methods

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Received : February 3, 2023	Revised : March 31, 2023	Accepted : April 1, 2023	Online : April 17, 2023

Abstract
The present study aims to obtain an analytic solution for the inhomogeneous Verhults equation using multiple expansion methods. This study identifies the external factors represented by the inhomogeneous term that determine optimal variable conditions for ecosystem population growth. The simulation involves scenarios that utilize constant growth rates, periodic growth rates, constant external factors, and periodic external factors. It is found that external factors increase population growth, whereas constant external factors prevent growth under saturation conditions. Periodic external factors cause fluctuations in the amplitude of growth regions. The present study will highlight and discuss the development and application of the solution.

Keywords: Environmental Factor; inhomogeneous Verhults equation, multiple scale expansion

1. INTRODUCTION

The analytical solution describing the evolution of population growth was proposed by Pierre Francois Verhulst, which has a sigmoid curve [1]. Ecologists and population scientists use population change models to describe the behavior of a population [2]. This model predicts how the population as a system experiences the dynamics of change. The model for population change generally consists of an exponential curve and a logistic curve. The main differences between the two models include three things: 1) Exponential growth is J-shaped while logistic growth is sigmoid (S-shaped), 2) Exponential growth only depends on population size but logistic growth depends on several factors including population size, competition, and a number of resources, and 3) Exponential growth applies to populations that do not have growth restrictions while logistical growth applies to each population with carrying capacity [3]-[6]. The exponential growth model describes certain data patterns that increase over time. The data graph reflects the exponential function and creates a J-shape. With respect to population change, exponential growth occurs when an unlimited amount of resources are available to the population. A clear example is the pest population which will grow exponentially if there is no limit to how much food the pest can eat from the garden. Another example is the current human population growing at an exponential rate. The logistic growth model describes data patterns where the growth rate gets smaller as the population approaches a certain maximum (known as the carrying capacity). The logistic growth graph is a sigmoid curve. With respect to population changes, logistic growth occurs when available resources are limited or when there is competition between animals. The pest population will increase until the introduction of pesticides and its carrying capacity allows the garden to thrive [7].

The behavior of the solution is exponential at the initial stage, followed by saturation, and finally, growth stops at maturity time. This equation has received a lot of attention from many researchers and has been shown to have various implications [8]. Although the equation has been known for a long time, many studies are still conducted to examine its general properties. Further investigation is carried out by considering the effect of the environment on growth [9][10].

The process of nature is irreversible, as manifested in an open system where there is an exchange of energy and matter between the system and the environment. The Verhulst or logistics equation is a closed system, making it interesting to consider the environmental effect on the system. In a closed system, the parameters of the models are assumed to be constant. However, in an open system, this assumption of constancy is not always true since the dynamics of such models can be influenced by environmental changes such as food availability, birth rates, disasters, etc. This means that the environmental effect should be taken into account in growth modeling. Recently, the effect of a slowly varying environment has also been discussed [10][11].

The Inhomogeneous Verhulst model has been used in determining pandemic curves, such as COVID-19 because it can determine the unpredictable peaks of second, third, and fourth-wave contamination. Taking into account the statistical inhomogeneity of age groups, it can be demonstrated quantitative understanding of the different behavior rules to flatten the COVID-19 pandemic curve and over time multi-peak periods simultaneously. The simulation is based on the Verhulst model with general analytical logistic equations for limited growth. The exponential growth for the first wave is faster than for the next wave [12]. This paper investigates the environmental effect on logistic processes with an external effect, i.e., the slowly varying effect of natural growth. The slowly varying effect is represented by an inhomogeneous term in the Verhulst equation. The multiple expansion methods will be used to solve the equation, and the behavior of the solution will be discussed.

2. MATERIALS AND METHODS

2.1. The Proposed Model

In this chapter, the inhomogeneity of the Verhults equation will be derived. It begins with the standard of the Verhulst equation that usually can be written in Eq.(1) [13].

$$\frac{\partial P}{\partial t} = rP\left(1-\frac{P}{K}\right) \qquad \textbf{(1)}$$

where $P(t)$ is called *the* population function, r defines the growth rate and K is called the carrying capacity (limitation of population, i.e., the maximum population density that the environment can carry). Eq. (1) have the basic structure in the form of $dP=dt= G(P) - R(P)$ where $G(P)$ is called the growth rate and $R(P)$ is the removal rate. In many cases, the growth rate and the carrying capacity are time-dependent. The quadratic term represents the competition between members of the same population for resources, i.e., intra- specific competition. The analytical solution of Eq. (1) with initial condition $P(0) = P_0$ and $\lim_{t\to\infty} P(t) = K$ is given by Goswami et al [14].

$$P(t) = \frac{P_0 e^{r(t-t_0)}}{1+\frac{P_0}{K}\left(e^{r(t-t_0)}-1\right)} \qquad \textbf{(2)}$$

Recently, the generalization of the Verhulst equation with environmental effect was done by adding the new term in Eq.(1) as shown in Eq.(3) [11].

$$\frac{\partial P}{\partial t} = rP\left(1-\frac{P}{K}\right)+f(P,t) \qquad \textbf{(3)}$$

where $f(P;\ t)$ is a function of P and t. Miskinis and Vasilliauskiene studied a special case of the environment called the influence of harvesting of the population wherethe Verhulst equation is given by Eq.(4) [10].

$$\frac{\partial P}{\partial t} = rP\left(1-\frac{P}{K}\right)+c(t)P \qquad \textbf{(4)}$$

where $c(t)$ is the harvesting population that is time-dependent. This is a homogeneous Verhulst equation with a variable coefficient (varying the growth rate). The analytic solution is given by Eq.(5) [10].

$$p(t) = P_0\, e^{\int_{t_0}^{t}\Theta(\tau)d\tau}\left[1+\frac{rP}{K}\int_{t_0}^{t} e^{\int_{t_0}^{t}\Theta(\tau)d\tau}\, d\tau\right]^{-1} \qquad \textbf{(5)}$$

where the new parameter $\Theta=r-c(t)$ represents differences between growth rate and harvesting. The variation of solution based on the harvesting function $c(t)$ can beobtained by solving the integral. It is notdifficult to show that for $c(t) = 0$, then the standard Verhuls equation was obtained. We modify Eq.(4) by taking into account the time dependent on the growth rate so that Eq.(4) yields Eq.(6).

$$\frac{\partial P}{\partial t} = \Theta(t)P - \Gamma(t)P^2 \qquad \textbf{(6)}$$

Where the new coefficient now are given by $\Theta(t) = r(t)-c(t)$ and $\Gamma(t)=r(t)/K$, respectively. It is clear, if the growth rate a constant then this yields the Miskinis and Vasilliauskiene model [10], i.e., Eq (4). The equation is the focus of the study.

2.2. An Analytic Solution

In this section, we discuss a slowly varying environment that is represented by the time-dependent coefficient in the inhomogeneous Verhulst equation. Assuming that the variation of the model parameters is slow relative to other quantities, we can apply the analytic multi-scaling technique to obtain solutions [15]. This method is expected to successfully represent the variation of the population over time. By assuming that the influence of the environment is weak, we can use the multiple time scale expansion method to find an analytical solution. The solution takes the form of Eq 7.,

$$P(t)=\sum_{n=0}^{N}\varepsilon^{n}P_{n}(t,\tau) \qquad \textbf{(7)}$$

where ϵ is a small parameter, $\tau = \epsilon t$ and the initial condition is the same as Eq (1). The differential operation satisfies the chain rule for example ($dg(t,\tau)/dt= \partial g/\partial t+\partial g/\partial\tau\times d\tau/dt$) the multiple scale expansion means that the nonlinear term is weak so that Eq. (6) become Eq.(8).

$$\frac{\partial P}{\partial t}=\Theta(t)P-\varepsilon\Gamma(t)P^{2} \qquad \textbf{(8)}$$

Substituting this solution Eq. (7) into Eq. (6) then we should solve the equation order by order. The equation for zero order is given by Eq.(9),

$$\frac{\partial P_0}{\partial t}=\Theta P_0 \qquad \textbf{(9)}$$

where P_0 is the zero order of the solution. Then the solution is given by Eq. 10.

$$P_0(t)=P_{00}\exp\left(\int_{t_0}^{t}\Theta(\xi)d\xi\right) \qquad \textbf{(10)}$$

Next, The first order is given by Eq. 11,

$$\frac{\partial P_1}{\partial t}=\Theta(t)P_1-\Gamma(t)P_0^2-\frac{\partial P_0}{\partial\tau} \qquad \textbf{(11)}$$

where P_1 is the first order of the solution This is an inhomogeneous coefficient variable differential equation of first order. The solution yield Eq. 12,

$$P_0(t)=\exp\left(\int_{t_0}^{t}\Theta(\xi)d\xi\right)\left[\exp\left(\int_{t_0}^{t}\Theta(\tau)d\tau\right)\left(-\Gamma(t)P_0^2-\frac{\partial P_0}{\partial t}\right)dt+P_{10}\right] \qquad \textbf{(12)}$$

due to ϵ being a small parameter then we can stop the calculation until the second order. Thus, the second order is given by Eq. 13,

$$\frac{\partial P_2}{\partial t}=\Theta(t)P_2-2\Gamma(t)P_0P_1-\frac{\partial P_1}{\partial\tau} \qquad \textbf{(13)}$$

The solution is given by Eq. 14,

$$P_0(t)=\exp\left(\int_{t_0}^{t}\Theta(\xi)d\xi\right)\left[\exp\left(\int_{t_0}^{t}\Theta(\tau)d\tau\right)\left(-2\Gamma(t)P_0P_1-\frac{\partial P_1}{\partial\tau}\right)d\tau+P_{20}\right] \qquad \textbf{(14)}$$

where P_{10} and P_{20} are an integration constant.

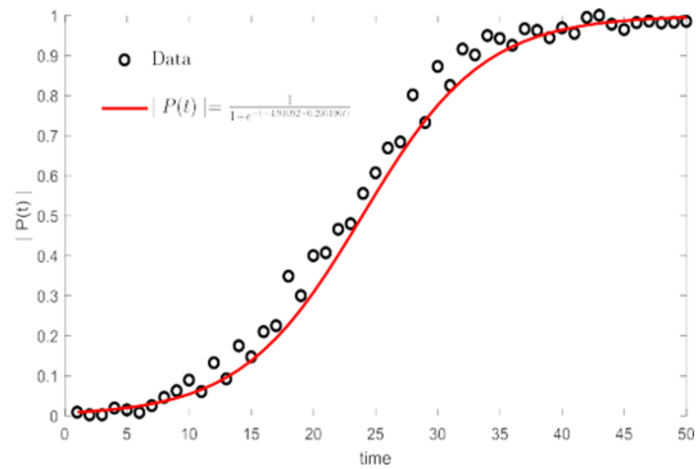

Figure 1. Regression between data (random sample) and solution of homogenous logistic equation.

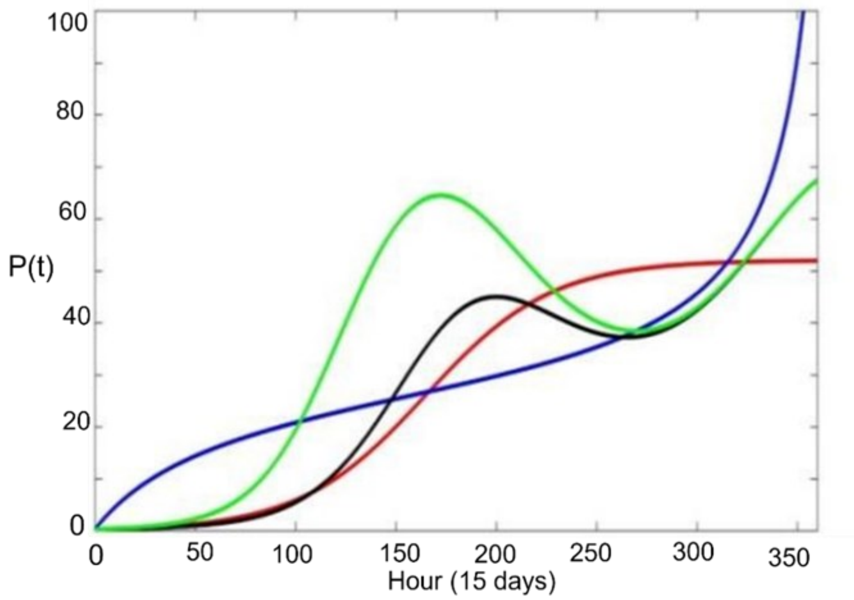

Figure 2. Simulation of Verhults solution with varying variable coefficients. The red color represents constant *r* and *c = 0*, the blue color represents constant *r* and *c*, the black color represents periodic *r* and constant *c*, and the green color represents periodic *r,* and *c*.

3. RESULTS AND DISCUSSIONS

We will begin with a simple system. In real-world data, we should use curve fitting to understand how the model behaves relative to the properties of the data. Since we do not have real-world data available, we will use simulated data instead. First, we normalize the data and the solution of the Logistic equation as follows Eq. 15,

$$P(t) = \frac{P(t)}{\max|P(t)|} = \frac{1}{1+e^{\alpha+\beta t}} \quad \textbf{(15)}$$

The data are generated by a random variable with 50 samples. Curve fitting between the data and logistic function is depicted in Fig. 1.

If the deviation of the data from the solution is small, then the system satisfies logistic homogeneity. Otherwise, if the variation is greater, then we fit the solution with the effect of the environment taken into account. To further study the environmental effect, let us use the same simulation parameters as in Miskinis et al. [10] where K = 52.0 mm, r = $3.19\times10^{-2}h^{-1}$, ϵ is about 0.1, and then initial condition P_0 is 0.27 mm and c=0.015 h^{-1}. We simulate the solution for four scenarios. In the first scenario, *c(t)* is zero, and *r(t)* is aconstant. In the second scenario, *c(t)* and *r(t)* are constant, and in the third scenario *c(t)* is a periodic function and *r(t)* is a constant. In the last scenario, *c (t)* and *r(t)* both are periodic functions with frequencies 3.3 π and 2.38 π, respectively. The simulation is depicted in Fig.2. The red color is a condition when *r is* constant and *c* is zero. This is nothing else than a standard Verhulst solution that is the same as Miskisnis et al. [10]. The blue color represents a simulation with a constant growth rate and constant external factors. This simulation shows that the positive constant external factor tends to increase growth. This result agreed with the previous works by Idlango et al. [9] and Tsoularis and Wallace [16]. The external factor will lead to growth, and the system will not become saturated. These results also show that the growth factor will dominate in the long run. When the growth rate tends to be periodic and the external factor remains constant, there is a fluctuation in the growth region, which eventually becomes constant after 15 days. This can be seen in the black color. A similar pattern can also be found when the growth rate and external factors are periodic. The fluctuations occur with a higher amplitude, which can be seen in the green color. They show the saturation condition for up to 350 days.

In general, logistic equations, including inhomogeneous ones, are typically solved numerically due to the availability of adequate numerical tools such as finite differences, among others, that are based on approximations [17].

However, an analytic solution is a closed-form mathematical expression that provides a complete and exact solution to a problem. It can offer a deeper understanding of the problem's behavior and properties and can also be used to develop theories, validate numerical methods, and guide experimental investigations. In some cases, we can easily calculate and predict how a system behaves when we apply external forces using simple computational tools that do not require large-capacity computers. However, for more complex cases, numerical studies may be necessary. For instance, the evolution of many species to ecosystem conditions with eutrophication and complex environment [18]-[20] and chaos phenomena [21] serve as an example. Despite its drawbacks, this method provides an in-depth look at the capabilities of logistic equations with their variations as one of the fundamental processes in nature.

4. CONCLUSIONS

The environmental effect in logistic processes, which is represented by variable coefficients in the growth rate and removal rate, has been investigated. The multiple expansion method was used to find an analytic solution of the variable coefficient of the inhomogeneous Varhults equation. We simulated scenarios with constant growth rate, periodic growth rate, constant external factor, and periodic external factor. It was found that the external factor tends to increase the growth of the population, whereas the constant external factor prevents growth in saturation conditions. Meanwhile, the periodic external factor tends to fluctuate the amplitude in the growth region. We will carry out simulations with ecological measurement data for the next research.

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Conflicts of Interest

The authors declare no conflict of interest.

ACKNOWLEDGEMENT

This research funded by DIPA 2018 the Ministry of Religious Affairs Republic Indonesia.

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